QUESTIONS:

- Springs and Masses
  - $\hookrightarrow$ How much damping is enough?
  - $\hookrightarrow$ What is natural frequency?
  - $\hookrightarrow$ How is amplitude related to initial data?
  - $\hookrightarrow$ How is amplitude related to the forcing term?

Damping the Springs

$$mx'' + px' + kx = 0$$

Facts:

(1) If $p > 0$ then the solutions always decay to zero.

(2) If p is *small*, then the solution oscillates for a long time before it dies out.

(3) If p is *large*, then the solution does not oscillate at all.

Where does the behavior change?

C.E. Tells All:

$$mx'' + px' + kx = 0 \quad \Longrightarrow \quad mr^2 + pr + k = 0$$

Look at the roots: $$r = \frac{-p \pm \sqrt{p^2 - 4mk}}{2m}$$

(i) If $p^2 - 4mk > 0$, then the roots are real and distinct

$\hookrightarrow$ decaying exponentials!

(ii) If $p^2 = 4mk$, then there is one root

$\hookrightarrow$ still decays away (but not as fast).

(iii) If $p^2 < 4mk$ then the roots are complex

$\hookrightarrow$ oscillations survive!

Examples:

$$x'' + px' + x = 0, \quad x(0) = 2,\ x'(0) = 0$$

- $p^2 < 4 \longrightarrow$ underdamping.
- $p^2 = 4 \longrightarrow$ critical damping.
- $p^2 > 4 \longrightarrow$ overdamping.

Solution:

$$\begin{aligned} x(t) \quad &= \quad \frac{\left(p^2 - 4 + \sqrt{p^2 - 4}\, p\right)}{p^2 - 4}\, \mathrm{e}^{\left(-1/2\, \left(p - \sqrt{p^2 - 4}\right)\, t\right)} \\ &\quad + \frac{\left(p^2 - 4 - \sqrt{p^2 - 4}\, p\right)}{p^2 - 4}\, \mathrm{e}^{\left(-1/2\, \left(p + \sqrt{p^2 - 4}\right)\, t\right)} \end{aligned}$$

Amplitude and Initial Data

$$x''(t) + 4x(t) = 0, \quad x(0) = x_i,\ x'(0) = v_i$$

General Solution:

$$x(t) = C_1 \sin(2t) + C_2 \cos(2t)$$

Solve for the constants:

$$\begin{aligned} x_i = x(0) &= C_1 \cdot 0 + C_2 \cdot 1 \\ v_i = x'(0) &= 2C_1 \cdot 1 - 2C_2 \cdot 0 \end{aligned}$$

Conclude: $$x(t) = \frac{v_i}{2} \sin(2t) + x_i \cos(2t)$$

A Little Calculus

What is the maximum value for $x(t) = C_1 \sin(2t) + C_2 \cos(2t)$?

$$\frac{dx}{dt} = 2C_1 \sin(2t) - 2C_2 \cos(2t) = 0 \iff \frac{\sin(2t)}{\cos(2t)} = \frac{C_1}{C_2}$$

"Critical Point" $$2t^* = \arctan\left(\frac{C_1}{C_2}\right)$$

For this value t^*,

$$x(t^*) = C_1 \sin\left(\arctan\left(\frac{C_1}{C_2}\right)\right) + C_2 \cos\left(\arctan\left(\frac{C_1}{C_2}\right)\right)$$

Aside:

Write the general solution as

$$x(t) = A\sin\Big(\omega_n t + \phi\Big)$$

where

$$A = \sqrt{x_i^2 + \left(\frac{v_i}{\omega_n}\right)^2}$$

and $\phi =???$

- This is the Phase Angle–Amplitude form of the solution.
- What is the period?

Amplitude and Forcing

Question: How does the amplitude of the solution depend on the frequency of the forcing term?

Example:

$$x'' + 4x = \cos(\omega t), \quad x(0) = 0,\ x'(0) = 0$$

Solve it:

$$x(t) = \frac{-1}{4-\omega^2}\cos(2t) + \frac{1}{4-\omega^2}\cos(\omega t)$$

Key: The amplitude becomes LARGE when $\omega \approx 2$

Key: When $\omega = 2$, this is NOT the solution.

What Happens when $\omega = \omega_n$?

Solve

$$x'' + \omega^2 x = F_0 \sin(\omega t), \quad x(0) = x_i,\ x'(0) = v_i$$

The particular solution will be

$$x_p(t) = At\sin(\omega t) + Bt\cos(\omega t)$$

Plug it in...

The Dirty Work:

$$\begin{aligned}
F_0 \sin(\omega t) &= \Big[2\omega A\cos(\omega t) - \omega^2 At\sin(\omega t) \\
&\qquad - 2\omega B\sin(\omega t) - \omega^2 Bt\cos(\omega t)\Big] \\
&\quad + \omega^2\Big[At\sin(\omega t) + Bt\cos(\omega t)\Big] \\
&= \Big[2\omega A\cos(\omega t) - 2\omega B\sin(\omega t)\Big]
\end{aligned}$$

$$x_p(t) = -\frac{F_0}{2\omega} t\cos(\omega t)$$

Minute Quiz:

How long will it take the solution

$$x'' + 4x = 10\sin(2t), \quad x(0) = 0,\ x'(0) = 0$$

to reach $+100$?

Approximate answer:

$$x_p(t) = -\frac{10}{4} t\cos(2t) \geq 100$$

Aside: Look at One Maple Solution

$$x''(t) + 4x(t) = \sin(2t)$$

$$\begin{aligned} x(\,t\,) = {} & \frac{1}{2}\cos(\,2\,t\,)\sin(\,t\,)\cos(\,t\,)^3 - \frac{1}{4}\cos(\,2\,t\,)\cos(\,t\,)\sin(\,t\,) \\ & - \frac{1}{4}\cos(\,2\,t\,)\arctan\left(\frac{\sin(\,t\,)}{\cos(\,t\,)}\right) + \frac{1}{2}\cos(\,t\,)^2\sin(\,2\,t\,) \\ & - \frac{1}{2}\cos(\,t\,)^4\sin(\,2\,t\,) + \_C1\cos(\,2\,t\,) + \_C2\sin(\,2\,t\,) \end{aligned}$$

Simplified a bit:

$$\begin{aligned} x(\,t\,) = {} & \frac{1}{8}\sin(\,2\,t\,) - \frac{1}{4}\cos(\,2\,t\,)\arctan\left(\frac{\sin(\,t\,)}{\cos(\,t\,)}\right) \\ & + \_C1\cos(\,2\,t\,) + \_C2\sin(\,2\,t\,) \end{aligned}$$

HW #9: Due Friday in Lecture

Sec 6.4: 4, 17, 20;

Sec 6.5: 2, 9, 17 ;

Sec 6.6: 6 ;

Project #2: