More Qualitative Analysis

- Start with a graph:

  $\hookrightarrow$ How much can you tell from a picture?

- Resonance for Damped Systems

  $\hookrightarrow$ What is the maximum amplitude?

- Steady States and Limiting Solutions

  $\hookrightarrow$ Life in the Phase Plane

Last Time: Amplitude and Forcing

Question: How does the amplitude of the solution depend on the frequency of the forcing term?

Example:

$$x'' + 4x = \cos(\omega t), \quad x(0) = 0,\ x'(0) = 0$$

Solve it:

$$x(t) = \frac{-1}{4-\omega^2}\cos(2t) + \frac{1}{4-\omega^2}\cos(\omega t)$$

Sketch a graph of amplitude versus frequency.

Example:

Attach a mass $m = 2$ to a spring with $k = 4$ and set it in motion from the equilibrium position with some (unknown) initial velocity v_i.

- If the measured amplitude of the motion is $12cm$, what is the initial velocity?
- If the period is measured to be 0.9 seconds, cwhat is the initial velocity?

What if...

- The amplitude decays from $12cm$ to $8cm$ after 2 minutes, can you determine the damping coefficient?

Damping the Springs

$$mx'' + px' + kx = F_0 \sin(\omega t)$$

Facts:

(1) For $p = 0$, *resonance* occurs at

$$\omega = \omega_n = \sqrt{\frac{k}{m}}$$

(2) If $p > 0$, the homogeneous solution is *transient*.

(3) If $p > 0$, resonance is still a problem.

Homogeneous Solution:

The Roots are negative: $r = \dfrac{-p \pm \sqrt{p^2 - 4mk}}{2m} < 0$

For example:

$$x_h(t) = e^{-\frac{p}{2m}t}\Big[C_1 \sin(\beta t) + C_2 \cos(\beta t)\Big]$$

where $\beta = \dfrac{1}{2m}\sqrt{4mk - p^2}$

Key:

$$\lim_{t\to\infty} x_h(t) = \lim_{t\to\infty} e^{-\frac{p}{2m}t}\Big[C_1 \sin(\beta t) + C_2 \cos(\beta t)\Big] = 0$$

Example:

$$x'' + \frac{1}{4}x' + x = 2\cos(3t), \quad x(0) = 2,\ x'(0) = 0$$

Solution:

$$\begin{aligned} x(t) &= e^{-\frac{1}{8}t}\left[C_1 \sin\left(\frac{3}{8}\sqrt{7}\, t\right) + C_2 \cos\left(\frac{3}{8}\sqrt{7}\, t\right)\right] \\ &\quad + \frac{24}{1033}\sin(3t) - \frac{256}{1033}\cos(3t) \end{aligned}$$

For “large” values of t:

$$e^{-\frac{1}{8}t} \le 0.001 \iff t \ge 8\ln(1000) \approx 55$$

There is a *limiting periodic solution.*

Limiting Periodic Solution

$$x'' + \frac{1}{4}x' + x = 2\cos(3t), \quad x(0) = 2,\ x'(0) = 0$$

- The equation has no steady state.
- All solutions converge to the same solution.

  Look at pictures

Look at the Phase Plane...

Graph

$$\begin{aligned} x(t) &= C_1 \sin(2t) + C_2 \cos(2t) \\ v(t) &= 2C_1 \cos(2t) - 2C_2 \sin(2t) \end{aligned}$$

Look at

$$\Big(x(t)\Big)^2 + \left(\frac{v(t)}{2}\right)^2$$

$$\begin{aligned} &= C_1^2 \sin^2(2t) + 2C_1C_2 \sin(2t)\cos(2t) + C_2^2 \cos^2(2t) \\ &\quad + C_1^2 \sin^2(2t) - 2C_1C_2 \sin(2t)\cos(2t) + C_2^2 \cos^2(2t) \\ &= \ldots \end{aligned}$$

Minute Quiz:

Define the Total Energy:

$$E(t) = \frac{1}{2}mv^2(t) + \frac{1}{2}kx^2(t)$$

Compute the derivative of the energy:

$$\begin{aligned} \frac{d}{dt}E(t) &= mv(t)v'(t) + kx(t)x'(t) \\ &= v(t)\Big[mx''(t) + kx(t)\Big] \\ &= x'(t)\Big[-px'(t)\Big] \\ &= -p\Big(x'(t)\Big)^2 \end{aligned}$$

HW #10: Due Tuesday in Conference

Sec 7.1: 3, 4, 6;

Sec 7.2: 2, 3, 13 ;

Sec 7.4: 8 ;

Project #2: Handouts available today at noon outside of my office.

You will be asked on Monday to choose your project.