TOPICS

- Pictures in the Phase Plane
  - $\hookrightarrow$ Springs and Masses (Again!)
  - $\hookrightarrow$ Competing Species
- Systems of Equations: The General Case

Phase Plane: Think Energy

$$E(t) = \frac{1}{2}mv^2(t) + \frac{1}{2}kx^2(t)$$

Use $mx'' + px' + kx = 0$ and differentiate the energy:

$$\begin{aligned}
\frac{d}{dt}E(t) &= mv(t)v'(t) + kx(t)x'(t) \\
&= v(t)\Big[mx''(t) + kx(t)\Big] \\
&= v(t)\Big[-px'(t)\Big] \\
&= -p\Big(v(t)\Big)^2
\end{aligned}$$

Implications:

- If $p = 0$, then $E(t)$ is constant.
- If $p = 0$, the solution are ellipses in the phase plane:

$$\frac{1}{2}mv^2(t) + \frac{1}{2}kx^2(t) = E(0)$$

- If $p > 0$, then $E(t)$ is decreasing.

  (The graphs in the phase plane are spirals...)

Pictures in the Phase Plane:

Example:

$$x'' + px' + x = 0, \quad x(0) = x_i,\ x'(0) = v_i$$

Turn it into a system

$$\begin{aligned} x'(t) &= v(t) \\ v'(t) &= -x(t) - pv(t) \end{aligned}$$

Look at solutions for different values of p.

Note: The Maple worksheets that generated all of the graphs in this lecture are available on the course homepage.

Remarks:

$$\begin{aligned} x'(t) &= v(t) \\ v'(t) &= -x(t) - pv(t) \end{aligned}$$

- $x'(t) = 0$ when $v(t) = 0$
- $v'(t) = 0$ when $v(t) = -\frac{1}{p}x(t)$

Limiting Periodic Solution

$$x'' + \frac{3}{2}x' + x = \sin(\omega t), \quad x(0) = x_i,\ x'(0) = v_i$$

- The equation has no steady state.
- The homogeneous solution is transient.
- All solutions converge to the same solution.

  Look at pictures

Example: Competing Species

$$\begin{aligned} x'(t) &= x\Big(3 - x - y\Big) \\ y'(t) &= y\left(2 - y - \frac{1}{2}x\right) \end{aligned}$$

Find the Steady States:

$$\begin{aligned} 0 &= x\Big(3 - x - y\Big) \\ 0 &= y\left(2 - y - \frac{1}{2}x\right) \end{aligned}$$

Four Solutions:

$$(x, y) \in \Big\{(0,0),\ (0,2),\ (3,0),\ (2,1)\Big\}$$

Question: Which of these are STABLE?

Linearize

Define "perturbations" from $(x_{ss}, y_{ss}) = (2, 1)$

$$\begin{aligned} p_1(t) &= x(t) - 2 \\ p_2(t) &= y(t) - 1 \end{aligned}$$

Plug in:

$$\begin{aligned} p_1' &= (p_1 + 2)\Big(3 - (p_1 + 2) - (p_2 + 1)\Big) \\ &= -2p_1 - 2p_2 - p_1(p_1 + p_2) \end{aligned}$$

$$\begin{aligned} p_2' &= (p_2 + 1)\Big(2 - (p_2 + 1) - \frac{1}{2}(p_1 + 2)\Big) \\ &= -\frac{1}{2}p_1 - p_2 - p_2((1/2)p_1 + p_2) \end{aligned}$$

Linearized Equation:

$$
\begin{aligned}
p_1' &= -2p_1 - 2p_2 \\
p_2' &= -\frac{1}{2}p_1 - p_2
\end{aligned}
$$

Question: How do you solve this system for p_1 and p_2?

Answer: Characteristic Equation!

Look for a solution pair

$$(p_1(t), p_2(t)) = \left(ae^{rt}, be^{rt}\right)$$

and solve for a, b, and r.

HW #10: Due Tomorrow in Conference

Sec 7.1: 3, 4, 6;

Sec 7.2: 2, 3, 13 ;

Sec 7.4: 8 ;

Project #2: Descriptions on the web page.