TOPICS

- Systems of Equations
  - $\hookrightarrow$ A Little Background
- When Coefficients are Constant...
  - $\hookrightarrow$ Characteristic Equation Works!

Announcements:

- Project Help Sessions
  - $\hookrightarrow$ Wednesday 5:00 — 6:30 in SH 203
  - $\hookrightarrow$ Thursday 9:00 — 10:30 in SH 306
- Test II:
  - $\hookrightarrow$ Next Wednesday in Lecture
  - $\hookrightarrow$ Re-Test Thursday at 3:30pm in Perrault

Tests will be graded by 4:30pm on Wednesday

No Conference on Thursday

System of ODEs

$$y' = f(t, y, z), \quad y(t_0) = y_0$$

$$z' = g(t, y, z), \quad z(t_0) = z_0$$

LINEAR System of ODEs

$$\begin{aligned} y' &= f_1(t)y + f_2(t)z \\ z' &= g_1(t)y + g_2(t)z \end{aligned}$$

This System is Homogeneous

Example:

$$\begin{aligned} \frac{dS}{dt} &= -\beta IS + \gamma R \\ \frac{dI}{dt} &= \beta IS - \nu I \\ \frac{dR}{dt} &= \nu I - \gamma R \end{aligned}$$

Note: This is a NON-linear system.

Draw a Picture

For LINEAR Systems

- General Solution requires two linearly independent solution pairs for the system.
- Today's equations: Homogeneous

Example: Constant Coefficients

$$\begin{aligned} y' &= -y + 3z \\ z' &= 3y - z \end{aligned}$$

Question: How do you solve this system for y and z?

Answer: Characteristic Equation!

Look for a solution pair

$$(y(t), z(t)) = \left(pe^{rt}, qe^{rt}\right)$$

and solve for p, q, and r.

Plug it in: Derive C.E.

$$\begin{aligned} rpe^{rt} &= -pe^{rt} + 3qe^{rt} \\ rqe^{rt} &= 3pe^{rt} - qe^{rt} \end{aligned}$$

$$\begin{array}{rcrcl} (r+1)p & - & 3q & = & 0 \\ -3p & + & (r+1)q & = & 0 \end{array}$$

Solve for p:

$$p\Big[(r+1)(r+1) - 9\Big] = 0$$

Either $p = 0$ <u>or</u> $(r+1)(r+1) - 9 = 0$

Characteristic Equation:

$$(r+1)(r+1) - 9 = 0 \implies r_1 = +2,\ r_2 = -4$$

You still have to find p and q

<u>For $r_1 = +2$:</u>

$$\begin{array}{rcrcl} (2+1)p & - & 3q & = & 0 \\ -3p & + & (2+1)q & = & 0 \end{array}$$

Both equations require $p = q$

Conclusion: you have one solution pair

$$\Big(y_1(t), z_1(t)\Big) = \Big(1e^{2t}, 1e^{2t}\Big) = \Big(e^{2t}, e^{2t}\Big)$$

Second Solution:

For $r_2 = -4$:

$$\begin{aligned} (-4+1)p \quad - \quad 3q &= 0 \\ -3p \quad + \quad (-4+1)q &= 0 \end{aligned}$$

Both equations require $p = -q$, so you have a second solution pair

$$\Big(y_2(t), z_2(t)\Big) = \Big(e^{-4t}, -e^{-4t}\Big)$$

Conclusion: you have the General Solution

$$\begin{aligned} \Big(y(t), z(t)\Big) &= C_1\Big(y_1(t), z_1(t)\Big) + C_2\Big(y_2(t), z_2(t)\Big) \\ &= \Big(C_1e^{2t} + C_2e^{-4t}, C_1e^{2t} - C_2e^{-4t}\Big) \end{aligned}$$

Loose Ends:

Are the solution pairs linearly independent?

The Wronskian works again:

$$W(t) = \det\begin{bmatrix} y_1(t) & y_2(t) \\ z_1(t) & z_2(t) \end{bmatrix} = y_1(t)z_2(t) - y_2(t)z_1(t)$$

If this is not zero for at least one t, the solution pairs are linearly independent and you really do have the general solution.

For this example,

$$W(t) = (e^{2t})(-e^{-4t}) - (e^{-4t})(e^{2t}) = -2e^{-2t}$$

Aside:

Once you have $y_1(t)$ and $y_2(t)$, use the ODE to find the z's

$$\text{ODE} \quad \left\{ \begin{array}{ccccc} y' & = & -y & + & 3z \\ z' & = & 3y & - & z \end{array} \right.$$

$$z(t) = \frac{1}{3}\Big(y'(t) + y(t)\Big)$$

$$y_1(t) = e^{2t} \implies$$

$$y_2(t) = e^{-4t} \implies$$

General Case:

$$\begin{array}{ccccc} y' & = & ay & + & bz \\ z' & = & cy & + & dz \end{array}$$

$$\begin{array}{rcrcl} (r-a)p & - & bq & = & 0 \\ -cp & + & (r-d)q & = & 0 \end{array}$$

CE:

$$\det \begin{bmatrix} (r-a) & -b \\ -c & (r-d) \end{bmatrix} = 0$$

Reality Check:

$$x'' + px' + kx = 0$$

Turn it into a system

$$\begin{aligned} x'(t) &= 0x(t) + 1v(t) \\ v'(t) &= -kx(t) - pv(t) \end{aligned}$$

New Characteristic Equation:

$$\det \begin{bmatrix} (0-r) & 1 \\ -k & (-p-r) \end{bmatrix} = -r(-p-r) + k = r^2 + pr + k = 0$$

Remarks:

- Three Things Can Happen:
  - $\hookrightarrow$ Distinct Real Roots
  - $\hookrightarrow$ Repeated Real Root
  - $\hookrightarrow$ Complex Conjugate Roots
- Three kinds of behavior ... *at least!*
- Friday: What about forcing terms?

HW #11: Due Friday in Lecture:

Sec 8.1: 8, 15

Sec 8.2: 1, 3, 16

Project #2: Descriptions on the web page.
Have you signed up with your PLA?