TOPICS

- You've ALMOST made it!
  - $\hookrightarrow$ TEST II: Wednesday at 9:00am
  - $\hookrightarrow$ Closed-book, No notes, Calculator OK
- REVIEW PROBLEMS

Announcements:

- Review Session for Test II:

  $\hookrightarrow$ Tuesday, March 4: 7:00pm in Perrault Hall

- Help Session for Project 2:

  $\hookrightarrow$ Tonight, 7:00pm in SH306

Guaranteed Problems:

1. Solve a second-order equation
   - $\hookrightarrow$ Homogeneous Solution & Particular Solution
   - $\hookrightarrow$ Initial Data
2. Analyze a spring-mass sytem
   - $\hookrightarrow$ Period, Frequency, Amplitude...
   - $\hookrightarrow$ Under, Over, Critical Damping...
   - $\hookrightarrow$ Resonance...
3. Solve a system of linear equations
   - $\hookrightarrow$ CE and Homogeneous Solution
   - $\hookrightarrow$ Find a Particular Solution
   - $\hookrightarrow$ Plots in the Phase Plane

Problem #1: Analyze a Linear Equation

$$y'' + 3y' + 2y = 3\cos(2t) - \sin(2t)$$

- Solve the homogeneous equation.
- Find the particular and the general solution.
- Add initial data.
- Find the transient component.
- Is there a limiting periodic solution?

Homogeneous Solution: CE

$$y'' + 3y' + 2y = 0$$

Characteristic Equation: $r^2 + 3r + 2 = (r + 2)(r + 1) = 0$

Distinct real roots: $y_1(t) = e^{-1t}$, $y_2(t) = e^{-2t}$

Test Linear Independence: $W(y_1, y_2) =???$

Undetermined Coefficients:

$$y_p(t) = A\sin(2t) + B\cos(2t)$$

Plug it in:

$$\begin{aligned} 3\cos(2t) - \sin(2t) &= y_p'' + 3y_p' + 2y_p \\ &= (-4A - 6B + 2A)\sin(2t) \\ &\quad +(-4B + 6A + 2B)\cos(2t) \end{aligned}$$

Sine Term: $-2A - 6B = -1$

Cosine Term: $6A - 2B = 3$

$$A = \frac{1}{2},\ B = 0 \implies y_p(t) = \frac{1}{2}\sin(2t)$$

Keep Going...

- General Solution: $y(t) = C_1 e^{-t} + C_2 e^{-2t} + \frac{1}{2}\sin(2t)$
- Find the <u>forced solution</u>: Set $y(0) = 0$, $y'(0) = 0$.

$$\begin{aligned} 0 &= y(0) = C_1 + C_2 + 0 \\ 0 &= y'(0) = -C_1 - 2C_2 + 1 \end{aligned}$$

Solution: $C_1 = -1$ and $C_2 = 1$, so the forced solution is

$$y(t) = -e^{-t} + e^{-2t} + \frac{1}{2}\sin(2t)$$

And Going...

- The transient part of the solution is $-e^{-t} + e^{-2t}$

  Solve $e^{-2t} - e^{-t} + 0.0001 = 0$

$$e^{-t} = \frac{1 \pm \sqrt{1 - 0.0004}}{2} \quad \Longrightarrow t \approx 9.210$$

- There is a limiting periodic solution: $\frac{1}{2}\sin(2t)$.

  The amplitude is $A = 1/2$ and the period is $T = \pi$.

Problem #2: Analyze a Spring-Mass System

A mass of 6kg is suspended from a spring that extended 0.1cm when a 1kg mass was attached. Pull the mass down 5cm and release it from rest.

- Basic Model: No damping, No forcing:

$$mx''(t) + kx(t) = 0, \quad x(0) = x_i,\ x'(0) = v_i.$$

- Determine some physical parameters...
- Add forcing...
- Add damping...

Analyze a Spring-Mass System

A mass of 6kg is suspended from a spring that extended 0.1cm when a 1kg mass was attached. Pull the mass down 5cm and release it from rest.

- Fix the parameters:

$$m = 6, \quad k = \frac{1 \cdot 9.8}{0.1} = 98, \quad x_i = -0.05, \quad v_i = 0$$

- Natural frequency $\omega_n = \sqrt{\dfrac{k}{m}} \approx 4.042$

  Period $T = \dfrac{2\pi}{\omega_n} \approx 1.555$

Keep Going ...

- Solve the IVP:

  General Solution $C_1 \sin(\omega_n t) + C_2 \cos(\omega_n t)$

  Use the initial data to get $x(t) = -0.05 \cos(\omega_n t)$

- Amplitude = 0.05 meters.

- Make the damping critical:

  $\hookrightarrow p^2 \leq 4mk \implies p \approx 48.5$

Bonus Problem:

What Conditions on a, b, c, d guarantee that the homogeneous solution will be transient?

$$\begin{array}{ccccccc} y' & = & ay & + & bz & + & f(t) \\ z' & = & cy & + & dz & + & g(t) \end{array}$$

CE: $r^2 - (a+d)y + (ad - bc) = 0$

Two roots: $r = \dfrac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$

These will be negative or have negative real part if

$$(a+d) < 0 \quad \text{and} \quad ad > bc$$

HW #12: Due Tuesday in Conference

Sec 8.3: 1, 3, 13

Project #2: Due Thursday at Noon in SH202A