TOPICS

- Two More Models
  - $\hookrightarrow$ Logistic Growth
  - $\hookrightarrow$ Heat Loss
- Linear Equations
  - $\hookrightarrow$ A Little Theory
  - $\hookrightarrow$ General Solution

Logistic Growth

Balance Equation:

$$[P(t+\Delta t) - P(t)] = [aP(t)]\,\Delta t - \left[s(P(t))^2\right]\Delta t$$

$$\text{Death Rate} = sP^2 \Longleftarrow \text{Competition}$$

ODE: Take a Limit

$$\frac{dP}{dt} = aP(t) - s(P(t))^2 = (a - sP(t)) \cdot P(t)$$

Sketch Some Solutions

$$\frac{dP}{dt} = (a - sP) \cdot P, \quad P(t_0) = P_i$$

√ $\quad \frac{dP}{dt} = 0$ when $P = 0$ or $a - sP = 0$

√ $\quad \frac{dP}{dt} > 0$ when $a - sP > 0$, or $P < a/s$

√ $\quad \frac{dP}{dt} < 0$ when $a - sP < 0$, or $P > a/s$

a/s is the *Carrying Capacity*

Heat Loss Model

$$\frac{dT}{dt} = -\frac{Ak}{cm}\left(T(t) - T_{\text{out}}\right)$$

Main Ingredients:

(1) Balance Law for Heat Energy Q

(2) Experimental Facts

Rate of Heat Energy Flow is Proportional to $T_{\text{in}} - T_{\text{out}}$

Heat Energy is Proportional to T

Linear Equations

Equations that you can SOLVE!

General Form:

$$a_1(x)\frac{dy}{dx} + a_0(x)y = f(x)$$

Independent Variable: x

Dependent Variable: $y \quad (= y(x))$

Coefficients: $a_1(x)$ and $a_0(x)$

Forcing Term: $f(x)$

Examples:

- $\dfrac{dP}{dt} = kP - H$

- $\dfrac{dT}{dt} = -\dfrac{Ak}{cm}\left(T(t) - T_{\mathrm{out}}\right)$

- $\dfrac{dP}{dt} = 0.012(1 - \sin(t))P - 0.65(1 + \cos(t))$

- $\dfrac{dP}{dt} = aP - sP^2$

Key Property: Superposition

If $y_1(x)$ solves

$$a_1(x)\frac{dy}{dx} + a_0(x)y = f(x)$$

and $y_2(x)$ solves

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

Then $y_1(x) + y_2(x)$ solves ???

Homogeneous Differential Equations

$$\longrightarrow \text{Zero is a solution.} \longleftarrow$$

When is a linear equation homogeneous?

Plug $y \equiv 0$ into

$$a_1(x)\frac{dy}{dx} + a_0(x)y = f(x)$$

Force the forcing term to be zero: $f(x) \equiv 0$:

Homogeneous Equation:

$$a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

Solve the Homogeneous Equation

$$a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

Separate and Integrate:

$$y(x) = C\exp\left(-\int \frac{a_0(x)}{a_1(x)}dx\right)$$

Define General Solution

$$a_1(x)\frac{dy}{dx} + a_0(x)y = f(x)$$

The General Solution is

$$y(x) = y_p(x) + Cy_h(x)$$

where

(1) $y_p(x)$ is *any* solution for the ODE.

(2) $y_h(x)$ is a non-trivial solution for the corresponding homogeneous equation.

Example:

$\frac{dP}{dt} = kP - H$ has general solution

$$P(t) = \frac{H}{k} + Ce^{kt}$$

Build the Solution

$$\frac{dT}{dt} + \frac{Ak}{cm}T(t) = \frac{Ak}{cm}T_{\text{out}}$$

(1) Solve $\frac{dT}{dt} + \frac{Ak}{cm}T(t) = 0$

$$T_h(t) = e^{-\frac{Ak}{cm}t}$$

(2) Find a particular solution:

$$T_p(t) = ???$$

HW #2: Due Thursday in Conference

Sec 2.1: 3d, 12a, 12b; **Sec 2.2: 2c, 4;**

• Go to your favorite non-Math book and bring back an ODE. You must provide:

(1) definitions of all variables

(2) brief explanation of application

(3) a complete reference (text and page)

HW #3: Due Monday in Lecture

Sec 2.3: 10, 11;

Sec 3.1: 36, 41;

Sec 3.2: 2, 3, 4;