TOPICS

- The Story So Far ...
- More About Linear Equations
  - $\hookrightarrow$ Definite/Indefinite
  - $\hookrightarrow$ Exact Equations and Integrating Factors
- Numerical Solutions
  - $\hookrightarrow$ Euler's Method

The Story So Far...

- A Few Models
  - $\hookrightarrow$ Free Fall, Populations, Heat Loss, Mixing...
- Some Basic Vocabulary
  - $\hookrightarrow$ ODE, IVP, Solution,...
- Analyzing Models
  - $\hookrightarrow$ Direction Fields, sketching solution curves
  - $\hookrightarrow$ Solving ODEs; Separate and Integrate,
  - $\hookrightarrow$ Linear $\Longrightarrow$ homogeneous + particular

Definite or Indefinite?

Solve

$$y' + g(x)y = f(x)$$

(1) find the homogeneous solution $y_h(x)$:

$$y_h(x) = \exp\left(-\int g(x)dx\right)$$

(2) find the particular solution $y_p(x)$:

$$y_p(x) = u(x)y_h(x) = \left[\int \frac{f(x)}{y_h(x)}dx\right] y_h(x)$$

Definite or Indefinite?

Solve

$$y' + g(x)y = f(x), \quad y(1) = \pi$$

(1) find the homogeneous solution $y_h(x)$:

$$y_h(x) = \exp\left(-\int_1^x g(z)dz\right)$$

(note that $y_h(1) = 1$)

(2) find the particular solution $y_p(x)$:

$$y_p(x) = u(x)y_h(x) = \left[\int_1^x \frac{f(z)}{y_h(z)}dz\right] y_h(x)$$

(note that $y_p(1) = 0$)

Finish it off:

The general solution is

$$y(x) = y_p(x) + Cy_h(x)$$

Now use the initial data:

$$\pi = y(1) = y_p(1) + Cy_h(1) = 0 + C \cdot 1 = C$$

This worked nicely because the integrals started at $x = 1$.

Example: An Exact Equation

The following equation is easy to solve:

$$\sin(t)\frac{dy}{dt} + \cos(t)y(t) = t^2$$

Key:

$$\frac{d}{dt}\Big[\sin(t) \cdot y(t)\Big] = \sin(t)\frac{dy}{dt} + \cos(t)y(t)$$

Exact derivative $\Longrightarrow$ the equation is really

$$\frac{d}{dt}\Big[\sin(t) \cdot y(t)\Big] = t^2$$

And you can just integrate!

So Just Integrate!

$$\int \frac{d}{dt}\left(\sin(t) \cdot y(t)\right) dt = \int t^2 dt$$

provides

$$\sin(t)y(t) = \frac{1}{3}t^3 + C$$

or

$$y(t) = \frac{1}{3}\csc(t)t^3 + C \cdot \csc(t)$$

Integrating Factor: Make an equation exact!

Start with a linear equation:

$$\frac{dy}{dt} + g(t)y(t) = f(t)$$

and multiply both sides by (an unknown function) $\mu(t)$:

$$\mu(t)\left[\frac{dy}{dt} + g(t)y(t)\right] = \mu(t) \cdot f(t)$$

Look at the ODE

$$\mu(t)\frac{dy}{dt} + \mu(t)g(t)y(t) = \mu(t)\frac{dy}{dt} + \frac{d\mu}{dt}y(t) = \frac{d}{dt}\left[\mu(t)y(t)\right]$$

Define $\mu(t)$ by:

$$\frac{d\mu}{dt} = \mu(t)g(t)$$

Another ODE: Just solve it!

$$\mu(t) = \exp\left(\int_0^t g(s)ds\right)$$

Call $\mu(t)$ an **Integrating Factor**

Example:

Use the integrating factor to solve:

$$y' + \cos(x)y = \pi\cos(x), \quad y(0) = 7.14$$

Integrating Factor:

$$\mu(x) = \exp\left(\int_0^x \cos(s)ds\right) = e^{\sin(x)}$$

Back to the ODE:

$$\frac{d}{dx}\left[y(x)e^{\sin(x)}\right] = \pi\cos(x)e^{\sin(x)}$$

and now integrate from 0 to x:

$$
\begin{aligned}
y(x)e^{\sin(x)} - 7.14 &= \int_0^x \pi \cos(z) e^{\sin(z)} dz \\
&= \pi e^{\sin(x)} - \pi e^{\sin(0)} \\
&= \pi e^{\sin(x)} - \pi
\end{aligned}
$$

So,

$$
\begin{aligned}
y(x) &= e^{-\sin(x)} \left[\pi e^{\sin(x)} - \pi + 7.14\right] \\
&= \pi + (7.14 - \pi)\, e^{-\sin(x)}
\end{aligned}
$$

Numerical Methods

The definition of the derivative

$$\frac{y(t+\Delta t) - y(t)}{\Delta t} \approx \frac{dy}{dt} = f(t, y)$$

a simple "one-step" formula:

$$y(t+\Delta t) \approx y(t) + f(t,y)\Delta t$$

In words:

$$\text{(New Value)} = \text{(Old Value)} + \text{(Slope)} \times \text{(time-step)}$$

Euler's Method:

Choose a stepsize Δt.

- $t_{k+1} = t_k + \Delta t$
- $y_{k+1} = y_k + f(t_k, y_k)\Delta t$

Start at the initial time t_0 and initial position y_0.

Examples:

Use five steps of Euler's method to approximate $P(50)$ for

$$P'(t) = 0.12P(t) - 0.011P^2(t), \quad P(0) = 1.0$$

- Step 0: $\Delta t = 10; \quad P_0 = 1$
- Step 1: $P_1 = P_0 + \left[0.12P_0 - 0.011P_0^2\right]10 = 2.09$
- Step 2: $P_2 = P_1 + \left[0.12P_1 - 0.011P_1^2\right]10 = 4.1175$
- Step 3: $P_3 = P_2 + \left[0.12P_2 - 0.011P_2^2\right]10 = 7.1936$
- Step 4: $P_4 = P_3 + \left[0.12P_3 - 0.011P_3^2\right]10 = 10.1336$
- Step 5: $P_5 = P_4 + \left[0.12P_4 - 0.011P_4^2\right]10 = 10.1980$

Conclude $P(50) \approx P_5 = 10.1980$

HW #4: Due Wednesday in Lecture

Sec 3.3: 6, 11, 12, 24, 32 ;

Sec 3.5: 5, 6, 9 ;

Project #1: Handout in Conference

Three students per team.

Sign up with PLA tomorrow.

Test #1: Next Wednesday (February 5) in Lecture