Exam given in term E97, July 24, 1997: **Solutions**

1. Solve the initial value problems

   (a) $x'' - 5x' + 6x = 0, \qquad x(0) = 0,\ x'(0) = -1.$

   The characteristic equation, $r^2 - 5r + 6 = 0$ has roots $2, 3$.
   The general solution is $x_g = C_1 e^{2t} + C_2 e^{3t}$.
   The initial conditions require $C_1 = 1$ and $C_2 = -1$. Therefore,

   $$x(t) = e^{2t} - e^{3t}$$

   (b) $x'' + 4x = 8\sin(2t), \qquad x(0) = 0,\ x'(0) = 0.$

   The general solution to the homogeneous solution is

   $$x(t) = C_1 \cos(2t) + C_2 \sin(2t).$$

   The particular solution has the form

   $$x_p(t) = (A\cos(2t) + B\sin(2t))t$$

   That gives

   $$x_p'' + 4x_p = -4A\sin(2t) + 4B\cos(2t)$$

   so $A = -2$ and $B = 0$.
   Therefore, the general solution to the nonhomogeneous equation is

   $$C_1 \cos(2t) + C_2 \sin(2t) - 2t\cos(2t)$$

   The initial conditions require $C_1 = 0$ and $C_2 = 1$.
   Therefore, the solution to the initial-value problem is

   $$x(t) = \sin(2t) - 2t\cos(2t)$$

2. For the differential equation $x'' + 6x' + 13x = 4\sin(2t)$

(a) Find the general solution to this equation.
The roots of the characteristic equation are $3 \pm 2i$.
The general solution to the homogeneous equation is

$$e^{-3t}(C_1\cos(3t) + C_2\sin(3t))$$

The particular solution has the form

$$x_p = A\sin(2t) + B\cos(2t)$$

The method of undetermined coefficients gives

$$A = \frac{4}{25}; \qquad B = \frac{16}{75}$$

The general solution is therefore

$$e^{-3t}(C_1\cos(3t) + C_2\sin(3t)) + \frac{4}{25}\sin(2t) + \frac{16}{75}\cos(2t)$$

(b) What part of the solution will become unimportant after a long time (the transient)? What part of the solution will be important after a long time (the limiting behavior)?
The transient is

$$e^{-3t}(C_1\cos(3t) + C_2\sin(3t))$$

The limiting solution is

$$\frac{4}{25}\sin(2t) + \frac{16}{75}\cos(2t)$$

(c) Find the amplitude and period of the limiting solution
The amplitude is

$$\sqrt{\left(\frac{4}{25}\right)^2 + \left(\frac{16}{75}\right)^2} = \frac{4}{15}$$

3. Write a differential equation that models a spring-mass system with mass 1 kg, damping coefficient 5 kg/s, and a spring constant 4 N/m. Is the system underdamped or overdamped? Using the same damping coefficient and spring constant, what mass will make the system critically damped?

   The differential equation is

$$x'' + 5x' + 4x = 0$$

   The characteristic equation, $r^2 + 5r + 4 = 0$, has real roots $-1, -5$ so the system is overdamped.

   The characteristic equation for general mass m is

$$mr^2 + 5r + 4 = 0,$$

   has discriminant $25 - 16m$. The system is critically damped when

$$m = \frac{25}{16}$$

4. For the first-order equation $y' = y^2 - 3y$, where are the equilibria? For each equilibrium, determine if it is stable or if it is unstable.

The equilibria are the roots of $y^2 - 3y$, or $y = 0$ and $y = 3$.
$y^2 - 3y > 0$ if $y > 3$ or $y < 0$, and $y^2 - 3y < 0$ if $0 < y < 3$.
Therefore solutions are

- increasing ($\nearrow$) if $y > 3$
- decreasing ($\searrow$) if $0 < y < 3$
- increasing ($\nearrow$) if $y < 0$

So the equilibrium at $y = 3$ is unstable and the equilibrium at $y = 0$ is stable.

Alternate solution to the question of stability:

The linearization about $y = 0$ is $y' = -3y$. The linearized equation is stable at $y = 0$, so the original equation is stable at $y = 0$.

Linearize about $y = 3$: Put $y = 3 + p$; then $y^2 - 3y = 3(y - 3) + (y - 3)^2$ so $p' = 3p + p^2$. The linearization around $y = 3$, or $p = 0$, is $p' = 3p$, which has an unstable equilibrium.

Therefore the original equation is unstable at $y = 3$.