TOPICS

- Qualitative Behavior

  ↪ Stability of Steady States

- Decomposing Solutions

  ↪ Free and Forced Solutions

- The Road Ahead

  ↪ Oscillations

Stability

- a **steady state** is any **constant solution**

- the **transient** component of a solution goes to zero as t goes to infinity

$$T' = -0.35\Big[T(t) - T_{\text{out}}\Big], \quad T(0) = T_i$$

leads to

$$T(t) = T_{\text{out}} + \Big[T_i - T_{\text{out}}\Big]e^{-0.35t}$$

$$\lim_{t\to\infty} T(t) = T_{\text{out}}$$

More Stability

Call a steady state y_{ss} **stable** if there is a $\beta > 0$ such that

$$\lim_{t\to\infty} y(t) = y_{\text{ss}} \quad \text{whenever } y_{\text{ss}} - \beta < y(0) < y_{\text{ss}} + \beta$$

Picture:

$$\frac{dy}{dt} = 3y - y^2$$

Logistic Equation

The carrying capacity is a stable steady state for

$$P' = aP - sP^2$$

Build the Error Equation

Define $\quad P(t) = \frac{a}{s} + \delta(t)$

Derive a new equation for the *error* $\delta(t)$:

$$\begin{aligned}
\frac{d}{dt}\left[\frac{a}{s} + \delta(t)\right] &= a\left[\frac{a}{s} + \delta(t)\right] - s\left[\frac{a}{s} + \delta(t)\right]^2 \\
&= \frac{a^2}{s} + a\delta(t) - s\left[\frac{a^2}{s^2} + 2\frac{a}{s}\delta(t) + \delta(t)^2\right] \\
&= a\delta(t) - 2a\delta(t) - s\delta(t)^2
\end{aligned}$$

Linearize the Error Equation

$$\frac{d}{dt}\delta(t) = -a\delta(t)$$

Solution: $\quad \delta(t) = \delta(0)e^{-at}$ goes to zero!

Exponential Decay for the Error!

Conclude:

$$P(t) = \frac{a}{s} + \delta(t) \text{ will converge to } \frac{a}{s} \ldots\ldots \text{(eventually)}$$

Free and Forced Response

Solve $\quad y'(x) + g(x)y(x) = f(x), \quad y(0) = y_i$

What part of the solution comes from the forcing function?

What part of the solution comes from the initial data?

$$y'(x) + g(x)y(x) = f(x), \quad y(0) = 0$$

$$y'(x) + g(x)y(x) = 0, \quad y(0) = y_i$$

Superposition Works!

Example:

$$y' + 4y = 100t, \quad y(0) = 200$$

Solution:

$$y_h(t) = e^{-4t}, \quad y_p(t) = 25t - \frac{25}{4}$$

$$y(t) = 25t - \frac{25}{4} + 206.25e^{-4t}$$

Build the Solution Again

Free Solution: $y' + 4y = 0, \quad y(0) = 200$

$$y_{\text{free}}(t) = \underline{\hspace{4cm}}$$

Forced Solution: $y' + 4y = 100t, \quad y(0) = 0$

$$y_{\text{forced}}(t) = \underline{\hspace{4cm}}$$

Do they add up?

Springs and Masses

Hooke's Law:

$$\left[\text{ force exerted by a sprint } = k \times \text{ extension of the spring }\right]$$

Add Newton's Law and a coordinate system:

$$mx''(t) = -kx(t)$$

General Solution:

$$x(t) = C_1 \sin\left(\sqrt{\frac{k}{m}}t\right) + C_2 \cos\left(\sqrt{\frac{k}{m}}t\right)$$

Characteristic Equations. . . Again

Look for solutions of the form $\sin(\omega t)$ and $\cos(\omega t)$:

$$m\Big(\sin(\omega t)\Big)'' + \ k\sin(\omega t) = 0$$

$$\big(-m\omega^2 + k\big)\sin(\omega t) = 0$$

Two roots:

$$\omega = \pm\sqrt{\frac{k}{m}}$$

What are the solutions?

Characteristic Equations. . . One more time!

Look for solutions of the form $e^{\lambda t}$.

$$m\Big(e^{\lambda t}\Big)'' + \ ke^{\lambda t}) = 0$$

$$\big(m\lambda^2 + k\big)\,e^{\lambda t}) = 0$$

Two *imaginary* roots:

$$\lambda = \pm i\sqrt{\frac{k}{m}}$$

What are the solutions?

HW #6: Due Tuesday in Conference

Sec 4.2: 2, 4, 19e, 19f ;

Sec 5.1: 1 ;

Re-Test #1: Today at 3:30pm in SL115

$\hookrightarrow$ Closed Book, Closed Notes, Calculator is Okay

$\hookrightarrow$ **Sign up** if you want to take the re-test.