TOPICS

- Coupled Systems (Phase Plane)

  $\hookrightarrow$ Predator and Prey

  $\hookrightarrow$ More Springs

- Solution Methods for Second-Order Problems

  $\hookrightarrow$ Linear Independence
  (you need <u>two</u> homogeneous solutions)

Coupled Systems

Basic Predator-Prey Model

$$
\begin{aligned}
F' &= -(d_F - \alpha R)F \\
R' &= +(b_R - \beta F)R
\end{aligned}
$$

Qualitative Analysis

- Steady State Population:

$$\begin{aligned} 0 &= -(d_F - \alpha R)F \\ 0 &= +(b_R - \beta F)R \end{aligned}$$

- Two solutions:

$$(F, R) = (0, 0), \quad (F, R) = \left(\frac{b_R}{\beta}, \frac{d_F}{\alpha}\right)$$

Qualitative Analysis

- $R < \dfrac{d_F}{\alpha} \Longrightarrow F' < 0; \qquad R > \dfrac{d_F}{\alpha} \Longrightarrow F' > 0$
- $F < \dfrac{b_R}{\beta} \Longrightarrow R' > 0; \qquad F > \dfrac{b_R}{\beta} \Longrightarrow R' < 0$

Sketch some solution curves...

you get oscillations

With Numbers...

$$d_F = 0.04, \quad \alpha = 0.0011, \quad b_R = 0.06, \quad \beta = 0.0009$$

Steady States:

$$F_{ss} = \frac{b_R}{\beta} \approx 66.67, \quad R_{ss} = \frac{d_F}{\alpha} \approx 36.36$$

Sketch some solution curves...

Spring Again!

Standard, unforced, undamped spring model:

$$mx'' + kx = 0, \quad x(0) = x_i, \; x'(0) = v_i$$

Rewrite it as a system:

$$\begin{aligned} \frac{dx}{dt} &= v(t) \\ \frac{dv}{dt} &= -\frac{k}{m}x(t) \end{aligned}$$

With initial data $x(0) = x_i, \; v(0) = v_i$

Graph solutions in the (x, v)–plane?

Spring Again!

Add Friction:

$$mx'' + px' + kx = 0, \quad x(0) = x_i,\ x'(0) = v_i$$

Rewrite it as a system:

$$\begin{aligned} \frac{dx}{dt} &= v(t) \\ \frac{dv}{dt} &= -\frac{p}{m}v(t) - \frac{k}{m}x(t) \end{aligned}$$

With initial data $x(0) = x_i,\ v(0) = v_i$

Graph solutions in the (x, v)–plane?

Linear Equations Again!

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = f(x)$$

Just as for first-order:

General Solution = Homogeneous + Particular

Troubles?

- The homogeneous solution is more complicated.
- You cannot just separate and integrate.
  (V of P and UC will still work!)

Homogeneous Solutions

- $x'' + 2x = 0$ has two solutions:

  $x_1(t) = \sin(\sqrt{2}t)$ and $x_2(t) = \cos(\sqrt{2}t)$

- $y'' - 9y = 0$ has two solutions:

  $y_1(x) = e^{3x}$ and $y_2(x) = e^{-3x}$

- $y'' + 2y' + y = 0$ has two solutions:

  $y_1(x) = e^{-x}$ and $y_2(x) = xe^{-x}$

Superposition Still Works!

Example: Solve an IVP

$$y''(x) - 9y(x) = 0, \quad y(0) = 10,\ y'(0) = 0$$

You need both y_1 and y_2: $y(x) = C_1e^{3x} + C_2e^{-3x}$

Data gives C_1 and C_2:

$$\begin{aligned} 10 = y(0) &= C_1 + C_2 \\ 0 = y'(0) &= 3C_1 - 3C_2 \end{aligned}$$

Solve it:

$$C_1 = 5, \quad C_2 = 5 \Longrightarrow y(x) = 5e^{3x} + 5e^{-3x}$$

Linear Independence

Call two functions **linearly independent**

on an interval $a \leq x \leq b$ if

$$C_1\, y_1(x) + C_2\, y_2(x) = 0 \quad \text{on } a \leq x \leq b$$

implies that

$$C_1 = C_2 = 0$$

Key: This is *a lot of equations* in 2 unknowns!

Example: $\sin(4t)$ and $\cos(4t)$ are L.I.D.

Solve a system—pick two values for t

$$\begin{aligned} 0 &= C_1 \sin(0) + C_2 \cos(0) \\ 0 &= C_1 \sin(\pi/2) + C_2 \cos(\pi/2) \end{aligned}$$

$\star$ This forces $C_1 = C_2 = 0$

There is another way!

Example: $\sin(4t)$ and $\cos(4t)$ are L.I.D.

Solve a system—Differentiate the equation

$$\begin{aligned} 0 &= C_1 \sin(4t) + C_2 \cos(4t) \\ 0 &= 4C_1 \cos(4t) - 4C_2 \sin(4t) \end{aligned}$$

This has a nonzero solution if and only if

$$0 = \det \begin{bmatrix} \sin(4t) & \cos(4t) \\ 4\cos(4t) & -4\sin(4t) \end{bmatrix}$$

⋆ But this is *never* zero.

⋆ Name it: the <u>Wronskian</u>

Summary: To solve

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = f(x)$$

(1) find two linearly independent homogeneous solutions

$$y_h(x) = C_1 y_1(x) + C_2 y_2(x)$$

(2) find the particular solution $y_p(x)$:

(3) combine the results:

$$y(x) = y_p(x) + C_1 y_1(x) + C_2 y_2(x)$$

HW #7: Due Friday in Lecture

Sec 5.1: 1, 9, 12 ;

Sec 5.3: 5, 6 ;

Sec 5.4: 1 ;

Project Evaluation in Conference Tomorrow

5 points for participation in the evaluation process.

(0 points if you skip conference)