TOPICS

- CE for Constant Coefficients

  $\hookrightarrow$ Three Things Happen...

  ...Solve a Quadratic Equation!

Summary: To solve

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = f(x)$$

(1) find two linearly independent homogeneous solutions

$$y_h(x) = C_1 y_1(x) + C_2 y_2(x)$$

(2) find the particular solution $y_p(x)$:

(3) combine the results:

$$y(x) = y_p(x) + C_1 y_1(x) + C_2 y_2(x)$$

Example:

Show that $y_1(x) = x^2$ and $y_2(x) = x^{-2}$ are linearly independent solutions for

$$x^2y'' + xy' - 4y = 0$$

Remarks:

- The Wronskian is a function of x.
- $W(x)$ satisfies its own ODE!

$$\frac{dW}{dx} + P(x)W(x) = 0$$

when y_1 and y_2 are solutions for

$$y'' + P(x)y' + Q(x)y = 0$$

Characteristic Equations

The key to the constant coefficient universe!

To solve

$$mx'' + px' + kx = 0$$

assume that

$$x(t) = e^{rt}$$

and forge ahead:

Characteristic Equation

$$mr^2 + pr + k = 0$$

Three Cases:

- Distinct Real Roots: r_1 and r_2
- Repeated Real Root: $r = r_1 = r_2$
- Complex Conjugate Roots:
  $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$

Distinct Roots $\Longrightarrow$ Life is Good

- $y'' - 9y = 0 \implies r^2 - 9 = 0$

  $r_1 = +3$, $r_2 = -3$ so $y_1(x) = e^{3x}$ and $y_2(x) = e^{-3x}$

- $x'' + 3x' + 2x = 0 \implies r^2 + 3r + 2 = 0$

  $r_1 = -1$, $r_2 = -2$ so $x_1(t) = e^{-t}$ $x_2(t) = e^{-2t}$

- Find an equation with solutions

$$y_1(x) = e^{3x} \text{ and } y_2(x) = e^{-(1/2)x}$$

Repeated Root $\Longrightarrow$ Life is Pretty Good

- $y'' + 4y' + 4y = 0 \implies r^2 + 4r + 4 = 0$

  $r_1 = -2$ and $r_2 = -2$ gives

  $y_1(x) = e^{-2x}$ and $y_2(x) =$???

To build another solution, assume that

$$y_2(x) = u(x)y_1(x)$$

Plug it in and cancel away:

Plugging...

The whole thing reduces to

$$u'' y_1(x) = 0$$

Test Independence

$$y_1(x) = e^{rx} \quad \text{and} \quad y_2(x) = xe^{rx} \quad \text{are L.I.D.}$$

Compute the Wronskian:

$$W(y_1, y_2) = \det \begin{bmatrix} e^{rx} & xe^{rx} \\ re^{rx} & e^{rx} + rxe^{rx} \end{bmatrix}$$

You get

$$W(y_1, y_2) = e^{2rx} + rxe^{2rx} - rxe^{2rx} = e^{2rx}$$

$\star$ and this is *never* zero $\star$

Complex Conjugate Roots $\Longrightarrow$ Life is Okay

Standard, unforced, undamped spring model:

$$mx'' + kx = 0 \implies mr^2 + k = 0$$

$$r_1 = +i\sqrt{\frac{k}{m}} \quad \text{and} \quad r_2 = -i\sqrt{\frac{k}{m}}$$

Example: Solve

$$y'' + 2y' + 2y = 0$$

CE: $r^2 + 2r + 2 = 0$ has roots

$$r = \frac{-2 \pm \sqrt{4 - 4(2)}}{2} = -1 \pm \frac{1}{2}\sqrt{-4} = -1 \pm i$$

Continue the Example:

You have L.I.D. solutions:

$$y_1(x) = e^{(-1+i)x} \quad \text{and} \quad y_2(x) = e^{(-1-i)x}$$

A more useful form:

$$e^{(-1+i)x} = e^{-x}e^{ix} = e^{-x}\Big[\cos(x) + i\sin(x)\Big]$$

$$e^{(-1-i)x} = e^{-x}e^{-ix} = e^{-x}\Big[\cos(x) - i\sin(x)\Big]$$

Compare Two General Solutions

$$y(x) = c_1 e^{(-1+i)x} + c_2 e^{(-1-i)x}$$

and

$$y(x) = e^{-x}\Big[A\sin(x) + B\cos(x)\Big]$$

Questions:

Are these the same solution?

How are the constants related?

Example: Solve an IVP

$$y''(x) + 2y'(x) + 2y(x) = 0, \quad y(0) = 10,\ y'(0) = 5$$

Solve for A and B

$$10 = y(0) = e^0 \Big[A\sin(0) + B\cos(0) \Big] = B$$

$$\begin{aligned} 5 &= y'(0) \\ &= -e^0 \Big[A\sin(0) + B\cos(0) \Big] + e^{-0} \Big[A\cos(0) - B\sin(0) \Big] \\ &= -10 + A \end{aligned}$$

So: $A = 15$ and $B = 10$

Remarks:

- Used

$$y'(x) = -e^{-x} \Big[A\sin(x) + B\cos(x) \Big] + e^{-x} \Big[A\cos(x) - B\sin(x) \Big]$$

- A and B are Real
- c_1 and c_2 are *complex*

$$c_1 = 5 + \frac{15}{2}i, \quad c_2 = 5 - \frac{15}{2}i$$

HW #8: Due Tuesday in Conference

Sec 6.1: 11–13 ;

Sec 6.2: 5, 10 ;

Sec 6.3: 1, 3, 5 ;

Project are being re-reviewed...