TOPICS

- Undetermined Coefficients
  - $\hookrightarrow$ Constant Coefficients
  - $\hookrightarrow$ Special Forcing Terms
- Numerical Methods
  - $\hookrightarrow$ Euler is Back!

Example:

Solve

$$x''(t) + 4x(t) = \sin(t)$$

- Step 1: $x_h(t) = c_1 \sin(2t) + c_2 \cos(2t)$

  $\hookrightarrow$ From the **CE:** $r^2 + 4 = 0 \implies r = \pm 2i$

- Step 2: Find a particular solution:

  Try $x_p(t) = A\sin(t) + B\cos(t)$

  Plug it in an solve for A and B.

Dirty Work:

- Step 2: Solve for A and B:

$$\Big[-A\sin(t) - B\cos(t)\Big] + 4\Big[A\sin(t) + B\cos(t)\Big] = \sin(t)$$

Two equations:

$$-A + 4A = 1, \quad -B + 4B = 0$$

$$A = \frac{1}{3}, \quad B = 0$$

- General Solution:

$$x(t) = \frac{1}{3}\sin(t) + C_1\sin(2t) + C_2\cos(2t)$$

Undetermined Coefficients Works

- $f(x) = e^{\lambda x}$ ....$y_p(x) = Ae^{\lambda x}$
- $f(t) = \sin(\omega t)$ ....$y_p(t) = A\sin(\omega t) + B\cos(\omega t)$
- $f(x) = x^n$ .....$y_p(x) = A_0 + A_1x^1 + A_2x^2 + \cdots + A_nx^n$

Remarks:

⋆ Watch for homogeneous solutions!

⋆ It works for <u>products</u> too!:

$$f(x) = e^{2x}\sin(5x) \implies y_p(x) = \underline{\hspace{4cm}}$$

Aside...

Bonus Problem: (Due Wednesday)

- What is the *annihilator?*
- What does it have to do with UC?

Example: Exponentials All Around

Solve $y'' - 9y = e^x, \quad y(0) = 0,\ y'(0) = 0$

- $y_h(x) = C_1 e^{3x} + C_2 e^{-3x}$
- $y_p(x) = Ae^x$ leads to $A = -\frac{1}{8}$
- General Solution: $y(x) = -\dfrac{1}{8}e^x + C_1 e^{3x} + C_2 e^{-3x}$
- Use the initial data to find C_1 and C_2.

$$C_1 = \frac{1}{12}, \quad C_2 = \frac{1}{24}$$

Repeated Root: Life is Pretty Good

Solve $y'' + 4y' + 4y = 5e^{-2x}$

- CE gives $r_1 = -2$ and $r_2 = -2$ so

  $$y_h(x) = C_1 e^{-2x} + C_2 x e^{-2x}$$

- UC says to look for $y_p(x) = ???$

  $y_p(x) = Ae^{-2x}$ won't work.

  $y_p(x) = Axe^{-2x}$ won't work.

  $y_p(x) = Ax^2e^{-2x}$ will work.

The Dirty Work:

Guess $y_p(x) = Ax^2e^{-2x}$ and plug it in

$$\begin{aligned}
f(x) = 5e^{-2x} &= \left[2Ae^{-2x} - 8Axe^{-2x} + 4Ax^2e^{-2x}\right] \\
&\quad +4\left[2Axe^{-2x} - 2Ax^2e^{-2x}\right] \\
&\quad +4\left[Ax^2e^{-2x}\right] \\
&= 2Ae^{-2x} \\
&\quad +\left[-8Axe^{-2x} + 8Axe^{-2x}\right] \\
&\quad +\left[4Ax^2e^{-2x} - 8Ax^2e^{-2x} + 4Ax^2e^{-2x}\right]
\end{aligned}$$

Summary:

The general solution for $y'' + 4y' + 4y = 5e^{-2x}$

$$y(x) = C_1 e^{-2x} + C_2 x e^{-2x} + \frac{5}{2} x^2 e^{-2x}$$

Questions:

- Can you choose the initial data so that the final solution contains only the particular solution?
- Does the solution have a transient component?
- Can you choose the initial data so that the final solution contains only the homogeneous solution?

Aside: Look at One Maple Solution

$$x''(t) + 4x(t) = \sin(2t)$$

$$\begin{aligned} x(t) = {} & \frac{1}{2}\cos(2\,t)\sin(t)\cos(t)^3 - \frac{1}{4}\cos(2\,t)\cos(t)\sin(t) \\ & - \frac{1}{4}\cos(2\,t)\arctan\left(\frac{\sin(t)}{\cos(t)}\right) + \frac{1}{2}\cos(t)^2\sin(2\,t) \\ & - \frac{1}{2}\cos(t)^4\sin(2\,t) + \_C1\cos(2\,t) + \_C2\sin(2\,t) \end{aligned}$$

Simplified a bit:

$$\begin{aligned} x(t) = {} & \frac{1}{8}\sin(2\,t) - \frac{1}{4}\cos(2\,t)\arctan\left(\frac{\sin(t)}{\cos(t)}\right) \\ & + \_C1\cos(2\,t) + \_C2\sin(2\,t) \end{aligned}$$

Euler's Method for Second-Order Equations

To approximate the solution to

$$mx''(t) + px'(t) + kx(t) = f(t), \quad x(0) = x_i,\ x'(0) = v_i$$

$\star$ Turn the equation into a system:

$$\begin{aligned} x'(t) &= v(t) \\ v'(t) &= \frac{1}{m}f(t) - \frac{p}{m}v(t) - \frac{k}{m}x(t) \end{aligned}$$

Use Euler as before:

$$\begin{aligned} x(t+h) &= x(t) + v(t)\cdot h \\ v(t+h) &= v(t) + \left[\frac{1}{m}f(t) - \frac{p}{m}v(t) - \frac{k}{m}x(t)\right]\cdot h \end{aligned}$$

Example:

To approximate the solution to

$$x''(t) + 4x(t) = 0, \quad x(0) = 1,\ x'(0) = 0$$

$\star$ Turn the equation into a system:

$$\begin{aligned} x'(t) &= v(t), \quad x(0) = 1 \\ v'(t) &= -4x(t), \quad v(0) = 0 \end{aligned}$$

And Approximate:

$$\begin{aligned} x(t+h) &= x(t) + v(t)\cdot\Delta t, \quad x(0) = 1 \\ v(t+h) &= v(t) - 4x(t)\cdot\Delta t, \quad v(0) = 0 \end{aligned}$$

Take a Few Steps:

$$
\begin{aligned}
x_{n+1} &= x_n + v_n \cdot \Delta t, \quad x_0 = 1 \\
v_{n+1} &= v_n - 4x_n \cdot \Delta t, \quad v_0 = 0
\end{aligned}
$$

$$
\left[\begin{array}{c|rrrrrr}
n & 0 & 1 & 2 & 3 & 4 & 5 \\
\hline
t & 0 & 0.10 & 0.20 & 0.30 & 0.40 & 0.50 \\
x & 1 & 1.00 & 0.96 & 0.88 & 0.76 & 0.68 \\
v & 0 & -0.40 & -0.80 & -1.18 & -1.54 & -1.84
\end{array}\right]
$$

HW #8: Due Tomorrow in Conference

Sec 6.1: 11–13 ;

Sec 6.2: 5, 10 ;

Sec 6.3: 1, 3, 5 ;

HW #9: Due Friday in Lecture

Sec 6.4: 4, 17, 20;

Sec 6.5: 2, 9, 17 ;

Sec 6.6: 6 ;