**TOPICS**

- Population Models
  - $\hookrightarrow$ Solve an IVP
  - $\hookrightarrow$ Analyze Data
  - $\hookrightarrow$ Balance Law
  - $\hookrightarrow$ Emigration and Harvesting
- HW #2: Due Thursday

**All mathematical models are wrong...**
**but some are more useful than others.**



**Solve an IVP**

$$P'(t) = 0.012P(t), \quad P(1900) = 76.09$$

Separate:

$$\frac{dP}{P} = 0.012dt$$

Integrate:

$$\ln|P(t)| = 0.012t + c \Longrightarrow P(t) = Ce^{0.012t}$$

What is $C$?

**Find the Constant**

$$76.09 = P(1900) = Ce^{0.012 \cdot 1900} = Ce^{22.8}$$

So

$$C = \frac{76.09}{e^{22.8}} \approx 9.537 \times 10^{-9}$$

Better Way:

$$P(t) = 76.09\, e^{0.012(t-1900)}$$



**Real Data: U.S. Population**

| Year | $P(t)$ | $D(t)$ | $D(t)/P(t)$ |
|---|---|---|---|
| 1790 | 3.93 | $\cdots$ | $\cdots$ |
| 1800 | 5.30 | 0.164 | $\cdots$ |
| 1810 | 7.22 | $\cdots$ | $\cdots$ |
| 1820 | 9.62 | $\cdots$ | $\cdots$ |
| 1830 | 12.90 | $\cdots$ | $\cdots$ |
| 1840 | 17.12 | $\cdots$ | $\cdots$ |
| 1850 | 23.26 | $\cdots$ | $\cdots$ |

**U.S. Population**

| Year | $P(t)$ | $D(t)$ | $D(t)/P(t)$ |
|---|---|---|---|
| 1790 | 3.93 | $\cdots$ | $\cdots$ |
| 1800 | 5.30 | 0.164 | 0.031 |
| 1810 | 7.22 | 0.216 | 0.030 |
| 1820 | 9.62 | 0.284 | 0.030 |
| 1830 | 12.90 | 0.375 | 0.029 |
| 1840 | 17.12 | 0.518 | 0.030 |
| 1850 | 23.26 | $\cdots$ | $\cdots$ |



**Exponential Growth**

$\hookrightarrow$ **Use Excel**

**Derive the Model**

$$D(t) = \frac{P(t+10) - P(t-10)}{20} \quad \text{and} \quad D(t) \approx \frac{dP}{dt}$$

**Model:**

$$\boxed{\frac{1}{P(t)} \cdot \frac{dP}{dt} = 0.030}$$



**Analyze the Model**

$$\frac{dP}{dt} = 0.03P(t)$$

General Solution:

$$P(t) = Ce^{(0.03 \cdot t)}$$

Use the Data from 1790 to get the Particular Solution:

$$P(t) = 3.93\ e^{0.03(t-1790)}$$

| Year | Actual | Model |
|---|---|---|
| 1790 | 3.93 | 3.93 |
| 1800 | 5.30 | 5.30 |
| 1810 | 7.22 | 7.16 |
| 1820 | 9.62 | 9.66 |
| 1830 | 12.90 | 13.05 |
| 1840 | 17.12 | 17.61 |
| 1850 | 23.26 | 23.77 |

Model Prediction: P(2000) = 2140.2 (million people)



**Alternate View: Balance Equation**

Net Change in time $\Delta t$ = (rate in)$\times\Delta t$ − (rate out)$\times\Delta t$

$\hookrightarrow$ Net Change $= P(t+\Delta t) - P(t)$

$\hookrightarrow$ Rate In $= k_{\text{in}} P(t)$

$\hookrightarrow$ Rate Out $= k_{\text{out}} P(t)$

## Transition to ODE

$$[P(t+\Delta t) - P(t)] = k_{\text{in}} P(t)\Delta t - k_{\text{out}} P(t)\Delta t$$

$$\frac{[P(t+\Delta t) - P(t)]}{\Delta t} = (k_{\text{in}} - k_{\text{out}})\, P(t)$$

Take the limit as $\Delta t \to 0$ to get an ODE:

$$\frac{dP}{dt} = kP(t)$$

where $k = k_{\text{in}} - k_{\text{out}}$



## General Remark:

Many ODE models are built this way:

Physical Law (Balance Equation)

+

Experimental Facts (Assumptions)

=

Approximate ODE

**Start Harvesting**

Same Balance Equation (almost):

$$P(t + \Delta t) - P(t) = [k_{\text{in}} P(t) - k_{\text{out}} P(t) - H]\,\Delta t$$

Divide by $\Delta t$ and let $\Delta t \to 0$:

$$\frac{dP}{dt} = kP(t) - H$$

Very different picture



**Find the Critical Population**

Harvest Model

$$\frac{dP}{dt} = kP(t) - H$$

$$\frac{dP}{dt} = 0 \iff kP(t) - H = 0$$

Claim:

If $P(0) \geq H/k$, the population survives.

**HW #2: Due Thursday in Conference**

**Sec 2.1: 3d, 7, 13a, 13b**; **Sec 2.2: 2c, 4**;

• Go to your favorite non-Math book and bring back an ODE. You must provide:

(1) definitions of all variables

(2) brief explanation of application

(3) a complete reference (text and page)

**Help Sessions:**

**Mondays, 6:30–8:00pm in SH304**

**Wednesdays, 8:00–9:30pm in SH308**