TOPICS

- The Story So Far ...
- More About Linear Equations
  - $\hookrightarrow$ Definite/Indefinite
  - $\hookrightarrow$ Exact Equations and Integrating Factors
- Numerical Solutions
  - $\hookrightarrow$ Euler's Method
  - $\hookrightarrow$ Heun's Method

The Story So Far...

- A Few Models
  - $\hookrightarrow$ Free Fall, Population, Heat Loss, Mixing...
- Some Basic Vocabulary
  - $\hookrightarrow$ ODE, IVP, Solution,...
- Analyzing Models
  - $\hookrightarrow$ Direction Fields, Sketching Solution Curves
  - $\hookrightarrow$ Solving ODEs; Separate and Integrate,
  - $\hookrightarrow$ Linear $\Longrightarrow$ homogeneous + particular = general

Recall...

Two steps in solving

$$y' + g(x)y = f(x)$$

(1) find the homogeneous solution $y_h(x)$:

$$y_h(x) = \exp\left(-\int g(x)dx\right)$$

(2) find the particular solution $y_p(x)$:

$$y_p(x) = u(x)y_h(x) = \left[\int \frac{f(x)}{y_h(x)}dx\right] y_h(x)$$

Trouble...

What do you do when you cannot evaluate the integrals?

$$\frac{dy}{dx} - 2xy = 1, \quad y(0) = 3$$

$$y_h(x) = \exp\left(-\int(-2x)dx\right) = \exp\left(x^2\right)$$

$$y_p(x) = u(x)y_h(x) = \left[\int \frac{1}{e^{x^2}}dx\right] e^{-x^2} = \left[\int e^{-x^2}dx\right] e^{x^2}$$

Stuck! (You cannot plug in $x = 0$!)

Fundamental Theorem of Calculus

$$\frac{d}{dx}\int_a^x g(z)dz = g(x)$$

for any constant a.

Application:

$$\begin{aligned}\frac{d}{dx}y_h(x) &= \frac{d}{dx}\exp\left(-\int_a^x g(z)dz\right)\\ &= -g(x)\exp\left(-\int_a^x g(z)dz\right)\\ &= -g(x)y_h(x)\end{aligned}$$

Go Definite

Solve

$$y' - 2xy = 1, \quad y(0) = 3$$

(1) find the homogeneous solution $y_h(x)$:

$$y_h(x) = \exp\left(\int_0^x 2z dz\right) = \exp\left(x^2\right) = e^{x^2}$$

(note that $y_h(0) = 1$)

(2) find the particular solution $y_p(x)$:

$$y_p(x) = u(x)y_h(x) = \left[\int_0^x \frac{1}{y_h(z)}dz\right] y_h(x) = \left[\int_0^x e^{-z^2}dz\right] e^{x^2}$$

(note that $y_p(0) = 0$)

Finish it off:

The general solution for

$$y' - 2xy = 1$$

is

$$y(x) = \left[\int_0^x e^{-z^2}\,dz\right] e^{x^2} + Ce^{x^2}$$

Use the initial data to find C:

$$3 = y(0) = y_p(0) + Cy_h(0) = 0 + C \cdot 1$$

so $C = 3$ and you have

$$y(x) = \left[\int_0^x e^{-z^2}\,dz\right] e^{x^2} + 3e^{x^2}$$

Example: An Exact Equation

The following equation is easy to solve:

$$\sin(t)\frac{dy}{dt} + \cos(t)y(t) = t^2$$

Key:

$$\frac{d}{dt}\left[\sin(t)\cdot y(t)\right] = \sin(t)\frac{dy}{dt} + \cos(t)y(t)$$

Exact derivative $\Longrightarrow$ the equation is really

$$\frac{d}{dt}\left[\sin(t)\cdot y(t)\right] = t^2$$

And you can just integrate!

So Just Integrate!

$$\int \frac{d}{dt}\left(\sin(t)\cdot y(t)\right)dt = \int t^2 dt$$

provides

$$\sin(t)y(t) = \frac{1}{3}t^3 + C$$

or

$$y(t) = \frac{1}{3}\csc(t)t^3 + C\cdot\csc(t)$$

Key Idea: You can make equations exact...

Numerical Methods

The definition of the derivative

$$\frac{y(t+\Delta t) - y(t)}{\Delta t} \approx \frac{dy}{dt} = f(t,y)$$

a simple “one-step” formula:

$$y(t+\Delta t) \approx y(t) + f(t,y)\Delta t$$

In words:

(New Value) = (Old Value) + (Slope) × (time-step)

Euler's Method:

Choose a stepsize Δt.

- $t_{k+1} = t_k + \Delta t$
- $y_{k+1} = y_k + f(t_k, y_k)\Delta t$

Start at the initial time t_0 and initial position y_0.

Examples:

Use five steps of Euler's method to approximate $P(50)$ for

$$P'(t) = 0.12P(t) - 0.011P^2(t), \quad P(0) = 1.0$$

- Step 0: $\Delta t = 10; \quad P_0 = 1$
- Step 1: $P_1 = P_0 + \left[0.12P_0 - 0.011P_0^2\right] 10 = 2.09$
- Step 2: $P_2 = P_1 + \left[0.12P_1 - 0.011P_1^2\right] 10 = 4.1175$
- Step 3: $P_3 = P_2 + \left[0.12P_2 - 0.011P_2^2\right] 10 = 7.1936$
- Step 4: $P_4 = P_3 + \left[0.12P_3 - 0.011P_3^2\right] 10 = 10.1336$
- Step 5: $P_5 = P_4 + \left[0.12P_4 - 0.011P_4^2\right] 10 = 10.9980$

Conclude $P(50) \approx P_5 = 10.9980$

Aside:

The exact solution for

$$P'(t) = 0.12P(t) - 0.011P^2(t), \quad P(0) = 1.0$$

is

$$P(t) = \frac{a}{s - Ce^{-at}} = \frac{0.12}{0.011 - (0.011 - 1)e^{-0.12t}}$$

Conclude $P(50) = 10.6476 \approx P_5 = 10.9980$

Examples: Easy with Excel!

Heun's Method: Average the Slopes

Choose a stepsize Δt.

- $t_{k+1} = t_k + \Delta t$
- $S_{\text{left}} = f(t_k, y_k)$
  $S_{\text{right}} = f(t_{k+1}, y_k + S_{\text{left}})$
- $y_{k+1} = y_k + \left(\dfrac{S_{\text{left}} + S_{\text{right}}}{2}\right) \Delta t$

Start at the initial time t_0 and initial position y_0.

HW #4: Due Wednesday in Lecture

Sec 3.3: 6, 11, 12, 24, 32 ;

Sec 3.5: 5, 6, 9 ;

Project #1: How is it growing?

Test #1: Next Wednesday (February 3) in Lecture