MA 2071 Linear Algebra Name: Solutions

Final a, E Term 2013 Section: ____________

Show all work needed to reach your answers.

1. (20 points)

(a) Please compute the determinant of the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 1 \\ 2 & -1 & 3 \end{bmatrix}$$

$$\begin{vmatrix} 1 & -1 & 2 \\ 0 & 2 & 1 \\ 2 & -1 & 3 \end{vmatrix} = 1\begin{vmatrix} 2 & 1 \\ -1 & 3 \end{vmatrix} + 2\begin{vmatrix} -1 & 2 \\ 2 & 1 \end{vmatrix}$$

$$= 7 + 2(-5)$$

$$= -3$$

determinant value: -3

(b) Is the above matrix singular or nonsingular? If it is singular, please explain why; if it is nonsingular, please compute its inverse.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 2 & -1 & 3 & 0 & 0 & 1 \end{array}\right] \rightarrow \begin{bmatrix} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -2 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -2 & 0 & 1 \\ 0 & 0 & 3 & 4 & 1 & -2 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & -1 & 0 & -5/3 & -2/3 & 4/3 \\ 0 & 1 & 0 & -2/3 & 1/3 & 1/3 \\ 0 & 0 & 1 & 4/3 & 1/3 & -2/3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & -7/3 & -1/3 & 5/3 \\ 0 & 1 & 0 & -2/3 & 1/3 & 1/3 \\ 0 & 0 & 1 & 4/3 & 1/3 & -2/3 \end{bmatrix}$$

$$\Longrightarrow A^{-1} = \frac{1}{3}\begin{bmatrix} -7 & -1 & 5 \\ -2 & 1 & 1 \\ 4 & 1 & -2 \end{bmatrix}$$

2. (6 points) Think carefully; compute little.

$$\begin{bmatrix} 1 & -1 & 2 & 6 & -1 \\ 0 & 2 & 1 & 5 & 4 \\ 0 & 0 & -1 & -2 & 7 \\ 0 & 0 & 0 & 4 & 11 \\ 0 & 0 & 0 & 0 & -3 \end{bmatrix}$$

(a) What are the eigenvalues of the matrix?

$\{1, 2, -1, 4, -3\}$

For a triangular matrix, the eVs are the entries on the diagonal.

(b) Why or why not is the above matrix diagonalizable?

Distinct eVs $\Rightarrow$ Diagonalizable

(c) What is the determinant of this matrix?

$det = 1 \cdot 2 \cdot (-1) \cdot 4 \cdot (-3) = \underline{24}$

3. (14 points) Please find a matrix A with eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ corresponding to the eigenvalue 3 and eigenvector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ corresponding to the eigenvalue 5.

$$A = PDP^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$$

$$P^{-1} = -\frac{1}{2} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 3 & 3 \\ 5 & -5 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 8 & -2 \\ -2 & 8 \end{bmatrix}$$

$A = \underline{\begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix}}$

4. (20 points) Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & -4 & 3 \\ 0 & 1 & -2 & 0 \\ 2 & -1 & 2 & 6 \\ -1 & 3 & -6 & -3 \end{bmatrix}$$

(a) Please find a basis, $\mathfrak{B}_1$, for Col(A), the column space of A.

$$\begin{bmatrix} 1 & 2 & -4 & 3 \\ 0 & 1 & -2 & 0 \\ 2 & -1 & 2 & 6 \\ -1 & 3 & -6 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & -4 & 3 \\ 0 & 1 & -2 & 0 \\ 0 & -5 & 10 & 0 \\ 0 & 5 & -10 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & -4 & 3 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

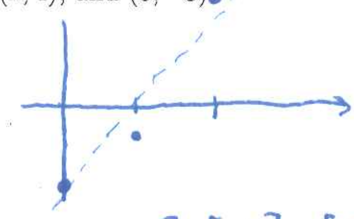
$$\mathcal{B}_1 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \\ 3 \end{bmatrix} \right\}$$

(b) Please find a basis, $\mathfrak{B}_2$, for Null(A), the null space of A.

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \mathcal{B}_2 = \left\{ \begin{bmatrix} 3 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} \right\}$$

5. (10 points) In terms of least squares, which line best fits that points $(1,-1)$, $(2,4)$, and $(0,-3)$?

$$\left.\begin{aligned} -1 &= m\cdot 1 + b \\ 4 &= m\cdot 2 + b \\ -3 &= m\cdot 0 + b \end{aligned}\right\} \Leftrightarrow \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} b \\ m \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ -3 \end{bmatrix}$$



$$A^TA = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix} \qquad A^Tb = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 4 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 7 \end{bmatrix}$$

So $(A^TA)x = A^Tb$ iff $\left[\begin{array}{cc|c} 3 & 3 & 0 \\ 3 & 5 & 7 \end{array}\right] \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -7/2 \\ 0 & 1 & 7/2 \end{bmatrix}$

$$\Rightarrow x = \begin{bmatrix} b \\ m \end{bmatrix} = \frac{7}{2}\begin{bmatrix} -1 \\ 1 \end{bmatrix} \Rightarrow \mathcal{L}: y = \frac{7}{2}x - \frac{7}{2}$$

6. (10 points) Are these vectors is linearly dependent or linearly independent? Please justify your answer.

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} \right\}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 4 \\ 3 & 1 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 3 \\ 0 & -2 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & -2 \end{bmatrix} \Rightarrow$$ Each column is a pivot column.

These vectors are L.I.

7. (12 points) Please use the Gram-Schmidt process to find an orthonormal basis for $\mathbb{R}^3$ using in order the basis vectors

$$\left\{ \begin{array}{c} X_1 \\ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \end{array}, \begin{array}{c} X_2 \\ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \end{array}, \begin{array}{c} X_3 \\ \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \end{array} \right\}$$

orthogonal basis:

$$V_1 = X_1 \qquad V_2 = X_2 - \frac{X_2 \cdot V_1}{\|V_1\|^2} V_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} - \frac{[1\ 1\ 2]\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}{1^2+0^2+1^2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} - \frac{3}{2}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1 \\ 1/2 \end{bmatrix} \xrightarrow{\text{Rescale}} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

$$V_3 = X_3 - \frac{X_3 \cdot V_1}{\|V_1\|^2} V_1 - \frac{X_3 \cdot V_2}{\|V_2\|^2} V_2 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} - \frac{[2\ 1\ -1]\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}{2}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{[2\ 1\ -1]\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}}{(-1)^2+2^2+1^2}\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \frac{1}{6}\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

$$\xrightarrow{\text{Rescale}} \begin{bmatrix} 8 \\ 8 \\ -8 \end{bmatrix} \xrightarrow{\text{Rescale}} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

orthonormal basis:

$$u_1 = \frac{V_1}{\|V_1\|} = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \qquad u_2 = \frac{V_2}{\|V_2\|} = \frac{\sqrt{6}}{6}\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \qquad u_3 = \frac{V_3}{\|V_3\|} = \frac{\sqrt{3}}{3}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

8. (8 points) Suppose $\mathcal{V}$ is a vector space over the reals. Suppose also that v_1 and v_2 are both vectors in $\mathcal{V}$, and $\alpha \in \mathbb{R}$.

(a) According to the definition of vector space,

$$\alpha(v_1 + v_2) = \underline{\alpha V_1 + \alpha V_2}$$

(b) Which of the following is **not** required by the definition of vector space:

✓ i. $v_1 + v_2 = v_2 + v_1$ Addition must be commutative.

✓ ii. $v_2 - v_2 = 0$ $\quad V_2 - V_2 = 1\cdot V_2 + (-1)\cdot V_2 = (1-1)\cdot V_2 = 0V_2 = \vec{0}$

X iii. $v_1 \cdot v_2 = 0$ if and only if $v_1 \perp v_2$

✓ iv. $1v_1 = v_1$ 1 must be the scalar identity.

Please explain your answer.

The dot product is not necessarily defined in a vector space, nor is there necessarily a definition of orthogonality.