MA 2071 Linear Algebra **Name:** Solutions

Final 1, E Term 2017 **Section:** ______

Show all work needed to reach your answers.

1. (8 points) For which value of h are these vectors orthogonal?

$$\begin{bmatrix} -3 \\ h \\ -7 \\ 4 \end{bmatrix}, \begin{bmatrix} -5 \\ 2 \\ -1 \\ -3 \end{bmatrix}$$

(+4) $15 + 2h + 7 - 12 = 0$ Dot Product

(+4) $h =$ -5

2. (12 points) Please solve the following linear system. Does a solution exist, and is it unique? Please state whether the system is consistent or inconsistent, and find all possible solutions. (+1)

$$\begin{array}{rcrcrcr} -x & + & 3y & + & 4z & = & -7 \\ 2x & - & 3y & - & 2z & = & 8 \\ -x & - & 3y & - & 8z & = & 5 \end{array}$$

(+1) Augmented Matrix (+2) (+2)

$$\left[\begin{array}{ccc|c} -1 & 3 & 4 & -7 \\ 2 & -3 & -2 & 8 \\ -1 & -3 & -8 & 5 \end{array}\right] \rightarrow \begin{bmatrix} 1 & -3 & -4 & 7 \\ 0 & 3 & 6 & -6 \\ 0 & -6 & -12 & 12 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & -4 & 7 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(+2)

$$\rightarrow \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} - t \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$$

The solution exists, but is not (+1) unique.

Any form of this solution (+3)

Solution:

$x_1 = 1 - 2t$

$x_2 = -2 - 2t$

$x_3 = t$

3. (10 points) Please evaluate this determinant without computing too much:
$$\begin{vmatrix} 3 & 1 & -2 & -3 \\ -2 & 0 & 4 & 0 \\ 1 & -1 & 0 & -3 \\ 0 & -1 & 0 & -3 \end{vmatrix}$$

$$= \begin{vmatrix} 3 & 1 & 2 & -3 \\ -2 & 0 & 4 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & -3 \end{vmatrix} = 1 \begin{vmatrix} 1 & 2 & -3 \\ 0 & 4 & 0 \\ -1 & 0 & -3 \end{vmatrix} = 4 \begin{vmatrix} 1 & -3 \\ -1 & -3 \end{vmatrix} = 4(-6)$$

(+3) (+2) (+2) (+3)

Determinant Value: -24

4. (10 points) For the matrix A below, please find a basis for the null space of A, Null(A).

(+2) (+2) (+2)

$$A = \begin{bmatrix} 1 & -2 & 1 & 1 \\ -1 & 3 & 0 & 2 \\ 2 & -3 & 3 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 1 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 1 & 1 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 1 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

free

$\vec{x} \in Null(A)$ iff $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3t \\ -t \\ t \\ 0 \end{bmatrix}$

(+2)

(+2) Any nonzero multiple of this vector is okay.

Basis: $\left\{ \begin{bmatrix} 3 \\ 1 \\ -1 \\ 0 \end{bmatrix} \right\}$

5. (12 points) Consider the matrix $A = \begin{bmatrix} 1 & 3 \\ 0 & -5 \\ -2 & -1 \end{bmatrix}$ and the vector $\mathbf{b} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$

For this A and $\mathbf{b}$, the system $A\mathbf{x} = \mathbf{b}$ is inconsistent. Please find the best least squares approximate solution, $\hat{\mathbf{x}}$.

$$A^T A = \begin{bmatrix} 1 & 0 & -2 \\ 3 & -5 & -1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & -5 \\ -2 & -1 \end{bmatrix} \overset{+4}{=} \begin{bmatrix} 5 & 5 \\ 5 & 35 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 1 & 0 & -2 \\ 3 & -5 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \overset{+4}{=} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

Solve $A^T A \hat{x} = A^T b$

$$\left[\begin{array}{cc|c} 5 & 5 & -1 \\ 5 & 35 & 2 \end{array}\right] \xrightarrow{+1} \begin{bmatrix} 5 & 5 & -1 \\ 0 & 30 & 3 \end{bmatrix} \xrightarrow{+2} \begin{bmatrix} 10 & 0 & -3 \\ 0 & 10 & 1 \end{bmatrix}$$

+1

$\hat{\mathbf{x}} = \begin{bmatrix} -3/10 \\ 1/10 \end{bmatrix}$

6. (12 points) Please find the 2×2 matrix with eigenvalues $\lambda_1 = 3$, $\lambda_2 = -1$ and eigenvectors $\mathbf{x}_1 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

$$\underline{A = PDP^{-1}}_{+2} = \underbrace{\overset{+2}{\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}} \overset{+2}{\begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}}} \overset{+3}{\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}}$$

$$P^{-1} = \frac{1}{\det(P)} \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$$

but $\det(P) = 1$

$$= \underbrace{\overset{+1}{\begin{bmatrix} 6 & -1 \\ 15 & -3 \end{bmatrix}} \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}}$$

$$\overset{+2}{=} \begin{bmatrix} 23 & -8 \\ 60 & -21 \end{bmatrix}$$

Matrix: $\begin{bmatrix} 23 & -8 \\ 60 & -21 \end{bmatrix}$

7. (10 points) Please write the vector $\begin{bmatrix} 1 \\ 9 \\ 2 \end{bmatrix}$ in terms of the basis $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \right\}$

We must find x_1, x_2 & x_3 such that

Vector Eqn

$$x_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 9 \\ 2 \end{bmatrix}$$

or Matrix Eqn

$$\begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 9 \\ 2 \end{bmatrix}$$

or Augmented Matrix +2

$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & 1 \\ 2 & 1 & 3 & 9 \\ 3 & 2 & 1 & 2 \end{array}\right] \xrightarrow{+2} \left[\begin{array}{ccc|c} 1 & 3 & 2 & 1 \\ 0 & -5 & -1 & 7 \\ 0 & -7 & -5 & -1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 3 & 2 & 1 \\ 0 & 5 & 1 & -7 \\ 0 & -2 & -4 & -8 \end{array}\right] \xrightarrow{+2} \left[\begin{array}{ccc|c} 1 & 3 & 2 & 1 \\ 0 & 1 & -7 & -23 \\ 0 & 0 & -18 & -54 \end{array}\right]$$

$$\xrightarrow{+2} \left[\begin{array}{ccc|c} 1 & 3 & 0 & -5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right] \xrightarrow{+1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

+1

Vector: $\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$

8. (10 points) For the matrix A below, please find the LU decomposition.

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ -4 & -6 & 1 & -4 & 1 \\ 0 & 4 & -2 & 3 & -2 \\ 3 & 1 & 1 & 18 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & -2 & 1 & 0 & 1 \\ 0 & 4 & -2 & 3 & -2 \\ 0 & -2 & 1 & 15 & -3 \end{bmatrix}$$

+2

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -4 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 3 & 1 & 5 & 1 \end{bmatrix}$$

+4

$$\to \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & -2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 15 & -4 \end{bmatrix}$$

+2

$$\to \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & -2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & -4 \end{bmatrix} = U$$

+1

+1

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -4 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 3 & 1 & 5 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & -2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & -4 \end{bmatrix}$$

9. (10 points) Fill in the blanks:

+2 each

- If A is an invertible matrix, the $\det(A^{-1}) =$ $\underline{\dfrac{1}{\det(A)}}$.
- If a matrix is invertible, then its eigenvalues are non zero.
- For any square matrix, the eigenvectors associated with distinct eigenvalues must be linearly independent.
- For a matrix A with n columns and m linearly independent rows ($m < n$), then $n - m$ is the dimension of Null(A).
- For an $m \times n$ matrix, the largest possible number of pivots is $\min\{n, m\}$

10. (6 points) Suppose that $A \in \mathcal{M}_{3,3}$ and that A^5 is the zero matrix. Explain why zero is the only possible eigenvalue of A.

+1

Suppose that $(\lambda, \vec{v})$ is an eV/e$\vec{v}$ pair. Then

+1 +1 +1

$$0 = [0]\vec{v} = A^5\vec{v} = A^4(A\vec{v}) = \lambda(A^4\vec{v}) = \lambda^5\vec{v}$$

+1 +1

Since $\vec{v} \neq \vec{0}$ (it's an e$\vec{v}$), $\lambda^5 = 0 \Rightarrow \lambda = 0$.

Thus zero is the only possible eV.