MA2431 Name: Solutions

Quiz 4 B Term, 2014

Show all work needed to reach your answers.

Consider the ODE system

$$\dot{x} = y + \varepsilon x \qquad \dot{y} = -x + \varepsilon y - x^2 y$$

#2-9

1. (10 points) Determine the type of bifurcation which occurs *at the origin* for this system as the control parameter ε passes through 0.

Linearize about (0,0)

$$\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \underbrace{\begin{bmatrix}\varepsilon & 1\\ -1 & \varepsilon\end{bmatrix}}_{A}\begin{bmatrix}x\\ y\end{bmatrix} \text{ (+2)} \Rightarrow \text{eVs: } \det(A-\lambda I) = (\varepsilon-\lambda)^2 + 1 = 0 \text{ (+2)}$$

$$\lambda^2 - 2\varepsilon\lambda + (\varepsilon^2+1) = 0$$

$$\lambda_\pm = \frac{2\varepsilon \pm \sqrt{4\varepsilon^2 - 4(\varepsilon^2+1)}}{2} = \varepsilon \pm i \text{ (+2)}$$

(+1) For $\varepsilon<0$, $Re(\lambda_\pm)<0$, so the solutions are stable spirals.

(+1) For $\varepsilon=0$, $\lambda_\pm = \pm i$, so the equilibrium point is singular (center)

(+1) For $\varepsilon>0$, $Re(\lambda_\pm)>0$, so the solutions are unstable spirals.

(+1) This combination means that this is a Hopf bifurcation (or a Hopf-like bifurcation)

2. (8 points) Please find the other equilibrium points (fixed points) for this system.

Set $\dot{x}=0=\dot{y}$ (+2) $0 = y+\varepsilon x \Rightarrow y = -\varepsilon x$

$$0 = -x + \varepsilon y - x^2 y \Leftrightarrow \varepsilon x^3 - \varepsilon^2 x - x = 0$$

$$\Leftrightarrow x(\varepsilon x^2 - (\varepsilon^2+1)) = 0 \text{ (+2)}$$

$$\Leftrightarrow x = \pm\sqrt{\varepsilon + 1/\varepsilon} \text{ (+2)}$$

So there are two other equilibrium points:

$$\left(\sqrt{\varepsilon + 1/\varepsilon}, -\sqrt{\varepsilon^3+\varepsilon}\right) \;\&\; \left(-\sqrt{\varepsilon + \tfrac{1}{\varepsilon}}, \sqrt{\varepsilon^3+\varepsilon}\right) \text{ (+2)}$$

3. (7 points) Please find the equation for $\dot{r}$ in terms of polar coordinates.

$$\left.\begin{aligned} x\dot{x} &= xy + \varepsilon x^2 \\ y\dot{y} &= -xy + \varepsilon y^2 - x^2y^2 \end{aligned}\right\} \text{ (+2)}$$

$$r\dot{r} = \varepsilon(x^2+y^2) - x^2y^2 \text{ (+2)}$$

$$= \varepsilon r^2 - r^4\cos^2\theta \sin^2\theta \text{ (+2)}$$

$$\Rightarrow \dot{r} = r(\varepsilon - r^2\cos^2\theta\sin^2\theta) \text{ (+1)}$$