MA3471 Name: Solutions

Final Exam (closed book, closed notes) **D Term, 2008**

1. (15 points) Consider the ODE $x'' + \sin(t)x' + 3x = 0$. Please write an equivalent 1^{st}-order system.

Let $y = x'$. Then (+5)

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -3 & -\sin t \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$ (+10)

2. (20 points) Please define/describe/state each of the following:

   (a) Lipschitz continuous w.r.t. x

   f is LC w.r.t. x iff $\exists\, K > 0$ s.t. $\|f(x,t) - f(y,t)\| \leq K\|x - y\|$ (+5)

   (b) autonomous ODE

   An ODE is autonomous iff $x' = f(x)$, (+5) i.e., there is no t in f (f is independent of t).

   (c) eigenfunction

   u is an eigenfunction of L iff $\exists\, \lambda \in \mathbb{C}$ s.t. $Lu = \lambda u$ (+5)

   (d) Hamiltonian system

   (+5) For $\begin{cases} x' = f(x,y) \\ y' = g(x,y) \end{cases}$ $\exists\, H(x,y)$ s.t. $f(x,y) = \frac{\partial H}{\partial y}(x,y)$ and $g(x,y) = -\frac{\partial H}{\partial x}(x,y)$

3. (10 points) What is the principle distinction between the hypotheses and conclusions of the two main existence theorems that we discussed in class, the Picard-Lindelof theorem and the Cauchy-Peano theorem.

The Picard-Lindelöf theorem requires that (+2) $f \in LC(Q)$ and in turn guarantees (+2) that $x' = f(t,x)$ has a unique (+2) solution. Cauchy-Peano only requires that $f \in C(Q)$ (+2) but only guarantees existence (+2).

4. (15 points) For the ODE $x' + \cos(t)x = 1$,

(a) Please find an integrating factor.

+5 $\mu(t) = e^{\int_{\pi/2}^{t} \cos(s)\,ds} = e^{\sin t}$

(b) Please use your integrating factor from above to write the left-hand side of the ODE as an exact derivative.

+5 $\left(x(t)e^{\sin t}\right)' = e^{\sin t}$

(c) Please find the general solution of this ODE (keeping in mind that it may not be possible to evaluate certain integrals).

+5 $x(t) = x_0 e^{-\sin t} + e^{-\sin t}\int_0^t e^{\sin(s)}\,ds$

5. (15 points) Can the Arzela-Ascoli theorem be applied to the sequence of functions $\{x_n(t) := \sin(nt)\}$, $0 \le t \le \pi$? Explain your answer.

No, because the sequence is certainly uniformly bounded (by 1), it is not equicontinuous. To see this, consider

$$|\sin(nt) - \sin(n\tau)|$$

for $t = \frac{\pi}{2n}$ and $\tau = 0$. Then $|\sin(nt) - \sin(n\tau)| = 1 > \epsilon = \frac{1}{2}$ even though $|t - \tau| = \frac{\pi}{2n} \to 0$ as $n \to +\infty$.

6. (15 points) Suppose that $x' = A(t)x$ with

$$A(t) = \begin{bmatrix} 2 & a(t) \\ b(t) & 4 \end{bmatrix}$$

where a and b are periodic with period π. If $\mu_1 = -1$ is one Floquet multiplier for this system, please find another Floquet multiplier. Also, what can be said about the stability of the trivial solution $x \equiv 0$?

By the trace thm $\mu_1 \mu_2 = e^{\int_0^\pi tr(A(t))dt} = e^{6\pi}$

So $\mu_2 = -e^{6\pi}$ and the trivial solution is unstable.

7. (10 points) Suppose $x = \psi(t)$ is an orbit (trajectory) of the ODE $x' = F(x)$ where $F \in C(\mathbb{R}^n)$ and $x : \mathbb{R} \to \mathbb{R}^n$. What condition on a function V guarantees that V is decreasing on the orbits of the ODE, and how do you know?

For V to be decreasing on an orbit,

$$\frac{d}{dt}\left(V(\psi(t))\right) < 0$$

So by the chain rule, this requires that

$$\frac{d}{dt}(V(\psi(t))) = \nabla V(\psi(t)) \cdot \psi'(t) = \nabla V(x) \cdot f(x) < 0$$

for $x = \psi(t)$. So if $\nabla V(x) \cdot f(x) < 0 \ \forall x$, then V must be decreasing.