MA3471 **Name:** Solutions Key

Final Exam (closed book, closed notes) **D Term, 2010**

Show all work needed to reach your answers. You may use any theorem we discussed in class, but cite any theorem you use.

1. (30 points) Suppose $A = \begin{bmatrix} 2 & 1 \\ 4 & -1 \end{bmatrix}$

$$\left.\begin{aligned} tr(A) &= 1 \\ det(A) &= -2-4=-6 \end{aligned}\right\} \Rightarrow eV(A)=\{-2,3\}$$

(a) Please find the general solution for $x' = Ax$.

For $\lambda_1 = -2$, $A - \lambda_1 I = A + 2I = \begin{bmatrix} 4 & 1 \\ 4 & 1 \end{bmatrix} \Rightarrow v_1 = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$ (or any multiple thereof)

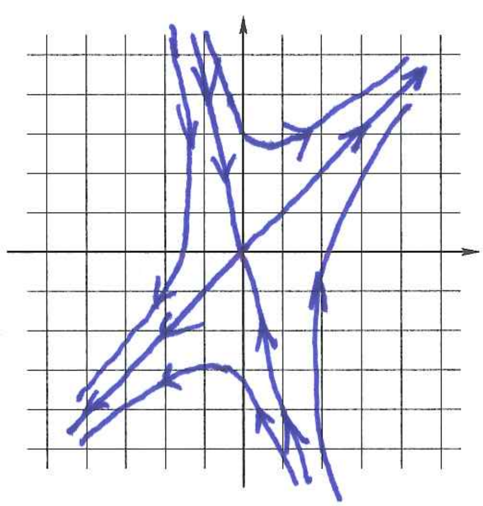
For $\lambda_2 = 3$, $A - \lambda_2 I = A - 3I = \begin{bmatrix} -1 & 1 \\ 4 & -4 \end{bmatrix} \Rightarrow v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

General Solution: $x(t) = c_1 e^{-2t}\begin{bmatrix} -1 \\ 4 \end{bmatrix} + c_2 e^{3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(b) Please find a fundamental matrix for $x' = Ax$.

Fundamental Matrix: $\Phi(t) = \begin{bmatrix} -e^{-2t} & e^{3t} \\ 4e^{-2t} & e^{3t} \end{bmatrix}$

(c) Please draw a rough sketch of ~~the~~ the phase plane diagram for this system.



(0,0) is a saddle point.

2. (15 points) Suppose that $x' = A(t)x$ with

$$A(t) = \begin{bmatrix} -2 & a(t) \\ b(t) & 1 \end{bmatrix}$$

where a and b are periodic with period 4π. If $\mu_1 = 1/4$ is one Floquet multiplier for this system, please find another Floquet multiplier. Also, what can be said about the stability of the trivial solution $x \equiv 0$?

+1 Integral

By the trace theorem $\mu_1 \mu_2 \overset{+2}{=} \mu_2/4 \overset{+1}{=} e^{\int_0^{4\pi} tr(A)\, ds} \overset{+1}{=} e^{-4\pi}$ +2 +1 +1

$\Rightarrow \mu_2 = 4e^{-4\pi} < 1$. Since $\mu_1 < 1$ too, +1 +1 +1

$X \equiv 0$ is asymptotically stable. +1 +2

3. (21 points) Consider the ODE $x' = -f(t, x)$ where $\forall\, t$, $f(t, x) > 0$ when $x > 0$; $f(t, x) < 0$ when $x < 0$; and $f(t, 0) = 0$. Suppose that the initial condition is $x(0) = 0$. So $x \equiv 0$ is *a* solution to this IVP.

(a) If f is Lipschitz continuous w.r.t. x, what theorem guarantees that $x \equiv 0$ is the only solution?

Picard-Lindelof theorem (+3)

(b) If f is *not* Lipschitz continuous w.r.t. x at $x = 0$, please explain why the solution $x \equiv 0$ is still unique.

Proof: To see that $x \equiv 0$ is the unique solution, suppose $\exists\ t_1 > 0$ such that $X(t_1) \neq 0$ (+1). Wolog (without loss of generality), suppose $X(t_1) > 0$ (+1). Since x is differentiable on $(0, t_1)$, it must also be continuous (+1) on this interval, so let t_0 be the largest value of $t < t_1$ such that $X(t_0) = 0$ (+1) and $X(t) > 0$ for $t_0 < t \le t_1$ (+1)

Notice that t_0 may be positive, but at a minimum, $t_0 = 0$.

Now by mean value theorem (+3) since x is differentiable on (t_0, t_1) (+2) (+1)

$x(t_1) =$ $X(t_1) \overset{+1}{-} X(t_0) = x'(c) \overset{+2}{}(t_1 - t_0)$

for some $c \in (t_0, t_1)$ (+1). But $x(t_1) > 0$ and $t_1 - t_0 > 0$, (+3) implying that $x'(c) > 0$ and contradicting that $x'(c) = -f(x(c), c) < 0$.

4. (24 points) Please define/describe/state each of the following:

(a) Hamiltonian system: $\exists\ H:\mathbb{R}^2 \to \mathbb{R}$ s.t. $x' = \partial_y H$ & $y' = -\partial_x H$ (+2)

So the system $\begin{cases} x' = f = \partial_y H \\ y' = g = -\partial_x H \end{cases}$ is Hamiltonian $\Rightarrow$ H is constant on trajectories. (+2) Also $\partial_x f = -\partial_y g$. (+1)

(b) integrating factor

If $x' + p(t)x = g(t)$, (+1) then the integrating factor is (+3) $\mu(t) = e^{\int p\, ds}$, ~~so that~~ $\mu(t)x' + p(t)\mu(t)x = (\mu(t)x)'$ (+2) is an exact derivative.

(c) strict Lyapunov function

For $x' = f(x)$, $V:\mathbb{R}^n \to \mathbb{R}$ is a strict Lyapunov function iff $V(x_0) = 0$ (+1), $V(x) > 0$ (+1) for $x \in N_\epsilon^\circ(x_0)$ $\exists\, \epsilon > 0$, and $\nabla V \cdot f < 0$ (+2)

The key point is that for any trajectory $x = \varphi(t)$, $\frac{d}{dt}(V(\varphi(t))) = \nabla V \cdot \varphi'(t) = \nabla V \cdot f < 0$

So V is strictly decreasing on solution trajectories. (+1) So x_0 is asymptotically stable. (+1)

(d) Strum-Liouville problem

Let $Lx := (p(t)x')' + q(t)x$. Then

$$\text{SLP}\begin{cases} Lx = -\lambda r(t) x \\ \alpha x(a) - \beta x'(a) = 0 \\ \gamma x(b) + \delta x'(b) = 0 \end{cases}$$ (+4)

provided $\alpha^2 + \beta^2 > 0$ and $\gamma^2 + \delta^2 > 0$ (+1)

The key point is that such an L is self-adjoint: $\langle Lx, y\rangle = \langle x, Ly\rangle$ (+1)

5. (10 points) Suppose that $x(t)$, $y(t)$ and $z(t)$ satisfy the system

$$\begin{aligned} x' &= yz \\ y' &= -2xz \\ z' &= xy \end{aligned}$$

What can be said about any solution to this system?
Please explain.

Let $V(x,y,z) = \rho^2 = x^2 + y^2 + z^2$. (+1) Since $f(x,y,z) = \langle yz, -2xz, xy\rangle$, $\nabla V \cdot f \equiv 0$ (+1) and hence by the Lyapunov stability theorem (+1) the equilibrium point (+1) $(0,0,0)$ is stable. (+1) But in fact even more can be said: Since $\frac{d}{dt}(V) = \frac{d}{dt}(\rho^2) = 2x\dot{x} + 2y\dot{y} + 2z\dot{z} = 2x(yz) + 2y(-2xz) + 2z(xy) \equiv 0$, (+2) ρ is constant for all (+1) trajectories. So all trajectories (+2) must stay on the sphere whose radius is given by the initial conditions:

$$\rho_0^2 = (x_0)^2 + (y_0)^2 + (z_0)^2$$