MA3471 Name: Solutions

Final Exam (closed book, closed notes) **D Term, 2012**

Show all work needed to reach your answers. You may use any theorem we discussed in class, but cite by name any theorem you use.

1. (10 points) For the ODE $x' + (4\cos t)x = \sin t$, please find an integrating factor $\mu(t)$ which can be used to make the left-hand-side an exact derivative.

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$$\mu(t) = e^{\int \frac{4\cos t}{1}\,dt}$$

Integrating Factor: $\mu(t) =$ $e^{4\sin t}$

2. (20 points) Suppose $A = \begin{bmatrix} 2 & 1 \\ -1 & a \end{bmatrix}$

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(a) For $a = -1$, what type of the equilibrium point is at the origin for the linear system $x' = Ax$.

$$\left.\begin{array}{l} tr(A) = 1 \\ det(A) = -1 \end{array}\right\} \Rightarrow \lambda(1-\lambda) = -1 \Rightarrow 0 = \lambda^2 - \lambda - 1 \Rightarrow \lambda_\pm = \frac{1 \pm \sqrt{5}}{2}$$

So $\lambda_- = \frac{1-\sqrt{5}}{2} < 0 < \frac{1+\sqrt{5}}{2} = \lambda_+$

$\Rightarrow$ The origin is a saddle node and thus unstable.

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(b) Is there a value of a such that origin is a *center* of $x' = Ax$? Please find such a value for a or explain why it does not exist.

$$0 = det(A - \lambda I) = \lambda^2 - (a+2)\lambda + (2a+1) = 0$$

$$\Rightarrow \lambda_{1,2} = \frac{(2+a) \pm \sqrt{(2+a)^2 - 4(1)(2a+1)}}{2}$$

For the origin to be a center $(2+a) = 0 \Rightarrow a = -2$

But the expression under the radical must also be negative, which it is <u>not</u> for $a = -2$. So the origin is never a center.

3. (20 points) Please define/describe/state each of the following:

5 (a) autonomous ODE

For $x \in \mathbb{R}^n$, the ODE $x' = f(t,x)$ is autonomous iff f is in fact independent of t. So $x' = f(x)$.

5 (b) sequence of equicontinuous functions $\{f_n(t)\}$

Given any $\epsilon > 0$, $(\forall n \in \mathbb{Z}^+)$ $(\exists \delta > 0)$ s.t. $\|f_n(t) - f_n(\tau)\| < \epsilon$ whenever $|t - \tau| < \delta$.

5 (c) Arzela-Ascoli theorem

Suppose that $F \subset \mathbb{R}^m$ is compact, and suppose that $\{f_k\}$ is a sequence of functions with $f_k : F \to \mathbb{R}^n$. If f_k are uniformly bounded and equicontinuous on F, then $\exists$ a subsequence $\{f_{k_j}\}$ which converges uniformly on F.

5 (d) limit cycle

A cycle is a nonconstant, periodic solution. A limit cycle is cycle which is approached by trajectories either as $t \to +\infty$ or as $t \to -\infty$.

4. (10 points) Suppose that $A(t)$ is π-periodic. Please show that if $\mu = -1$ is a Floquet multiplier for $x' = A(t)x$, then there is a nontrivial solution with period 2π.

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By an unnamed theorem in Floquet theory (Thm 2.73), if A is π-periodic and -1 is a Floquet multiplier, then $\exists$ a solution of $x' = A(t)x$ s.t. $x(t+\pi) = -x(t)$. So $x(t+2\pi) = -x(t+\pi) = x(t) \Rightarrow$ This nontrivial solution is 2π periodic.

5. (15 points) Consider the BVP: $\epsilon y'' + yy' + y^3 = 0$, $y(0) = 1$, $y(1) = 3$.

(a) Please use dominant balance to find the correct inner variable $X = x/\epsilon^\alpha$ for a boundary layer region near $x = 0$.

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If $X := \frac{x}{\epsilon^\alpha}$, then $\frac{d}{dx} = \frac{dX}{dx}\frac{d}{dX} = \epsilon^{-\alpha}\frac{d}{dX}$.

Let $Y(X) = Y\left(\frac{x}{\epsilon^\alpha}\right) := y(x)$. Then

$$\epsilon^{1-2\alpha} Y'' + \epsilon^{-\alpha} Y Y' + Y^3 = 0$$

Balancing the first and second terms: $1 - 2\alpha = -\alpha \Rightarrow \alpha = 1$ Good

Balancing the first and third terms: $1 - 2\alpha = 0 \Rightarrow \alpha = 1/2$ Bad because the second term is then $\epsilon^{-1/2}$

So $X = \frac{x}{\epsilon}$ is the correct inner variable.

(b) Please find the zeroth-order inner equation.

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The zeroth-order inner equation is then based on the first and second terms:

$$Y_0'' + Y_0 Y_0' = 0$$

6. (15 points) Please find constants α and β so that $V(x,y) := \alpha x^2 + \beta y^2$ defines a Liapunov function for the system

$$
\begin{aligned}
x' &= -x - 5y \\
y' &= 3x - y^3
\end{aligned}
$$

Since $V(0,0) = 0$ and $V > 0$ for $(x,y) \neq (0,0)$, one needs that

5 $\nabla V(x,y) \cdot (f(x,y), g(x,y)) = (2\alpha x, 2\beta y) \cdot (-x-5y, 3x-y^3) \leq 0$

for $(x,y) \neq (0,0)$ (at least near the origin). So one needs

$$-2\alpha x^2 - 10\alpha xy + 6\beta xy - 2\beta y^4 \leq 0.$$

This inequality holds if $\alpha = 3$, $\beta = 5$ (among other possibilities). So 5

$V(x,y) = 3x^2 + 5y^2$ is a Liapunov function. rest 5

7. (10 points) For an interval $[a,b] \in \mathbb{R}$, suppose that $p, q \in C[a,b]$. Suppose that x_1 and x_2 are both solutions of $(p(t)x')' + q(t)x = h(t)$ satisfying $x(a) = \cancel{1}$, $x(b) = \cancel{-1}$. What can be said about the Wronskian $W[x_1, x_2]$? Please explain your answer. Should be zero!

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Let $Lx := (p(t)x')' + q(t)x$. Then by the Lagrange identity,

$$h(t)(x_1 - x_2) = x_1 Lx_2 - x_2 Lx_1 = \frac{d}{dt}\left(p(t) W[x_1, x_2]\right)$$

Integrating:

$$\int_a^t h(s)(x_1 - x_2)\, ds = p(t) W[x_1, x_2] - p(a)\underbrace{\left(x_1(a) x_2'(a) - x_2(a) x_1'(a)\right)}_{0}$$

So provided $p(t) > 0$,

$$W[x_1, x_2] = \frac{1}{p(t)} \int_a^t h(s)(x_1 - x_2)\, ds$$

In particular, if $h(s) \equiv 0$, then $W[x_1, x_2] \equiv 0$.