MA3471 Name: Solutions

Quiz 2 D Term, 2010

Show all work needed to reach your answers.

1. (10 points) Please decide which of the following is a Hamiltonian system. If a systems is Hamiltonian, please find the Hamiltonian function H.

(a)
$$\begin{aligned} x' &= 8xy + x\sin(xy) \\ y' &= -x - 4y^2 - y\sin(xy) \end{aligned}$$

$\frac{\partial f}{\partial x} = 8y + \sin(xy) + xy\cos(xy)$ +1

$\frac{\partial g}{\partial y} = -8y - \sin(xy) - xy\cos(xy)$

Since $\frac{\partial f}{\partial x} = -\frac{\partial g}{\partial y}$, this system is Hamiltonian. +1

So $\frac{\partial H}{\partial y}(x,y) = 8xy + x\sin(xy)$ +1 $\Rightarrow$ $H(x,y) = 4xy^2 - \cos(xy) + C(x)$ +1

$\Rightarrow \frac{\partial H}{\partial x}(x,y) = 4y^2 + y\sin(xy) + C'(x) = -g(x,y) = -(-x - 4y^2 - y\sin(xy))$ +1 +1

$\Rightarrow C'(x) = x$ +1 $\Rightarrow C(x) = \frac{1}{2}x^2$ +1

Thus $H(x,y) = 4xy^2 - \cos(xy) + \frac{1}{2}x^2$

(b)
$$\begin{aligned} x' &= 2x + y \\ y' &= 8x + 2y \end{aligned}$$

$\frac{\partial f}{\partial x} = 2$ +1 $\frac{\partial g}{\partial y} = 2$

So $\frac{\partial f}{\partial x} \neq -\frac{\partial g}{\partial y} \Rightarrow$ not Hamiltonian. +1

2. (10 points) Let (0,0) be an equilibrium point for the ODE system $x' = f(x,y)$, $y' = g(x,y)$.

(a) Which of the following could be a Liapunov function V_k for this system? (Circle each potential Liapunov function; cross out any V_k that *can not* be Liapunov.)

+4

(1) $V_1(x,y) = x^2 + 8xy + y^2$ (2) $V_2(x,y) = x^8 + 5y^4 + \cos x - 1$ (3) $V_3(x,y) = 6x^2 + \cos(y)$

$V(x,-x) = -6x^2 < 0$ $V(x,y) < 0$ for $0 < x \ll 1$ $V(0,0) \neq 0$

(b) Suppose $x = \varphi(t)$, $y = \psi(t)$ is a solution trajectory for this system. If V is a *strict* Liapunov function for the system, please show that V is strictly decreasing along this trajectory. (Hint: chain rule.)

$$\frac{d}{dt}\left(V(\varphi(t),\psi(t))\right) = \nabla V(\varphi(t),\psi(t)) \cdot (\varphi'(t), \psi'(t))$$ +2

Chain Rule

$$= \nabla V(\varphi(t),\psi(t)) \cdot (f(\varphi(t),\psi(t)), g(\varphi(t),\psi(t)))$$ +1

the ODE

$$< 0$$ +2

Liapunov inequality

So V is strictly decreasing. +1