MA3471 Name: Solutions

Quiz 3 **D Term, 2012**

Show all work needed to reach your answers. Please cite by name any theorem you use.

1. (10 points) Consider the system of two ODEs

$$\begin{aligned} x' &= x + y - x(3x^2 + y^2) \\ y' &= -x + y - y(3x^2 + y^2) \end{aligned}$$

   (a) Please convert this system to polar coordinates. Hint: $r^2 = x^2 + y^2 \Rightarrow rr' = xx' + yy'$

$rr' = x(x+\cancel{y} - x(3x^2+y^2)) + y(-\cancel{x}+y-y(3x^2+y^2))$

$= r^2 - r^2(r^2 + 2x^2) = r^2 - r^4(1+2\cos^2\theta)$

$\Rightarrow r' = r - r^3(1+2\cos^2\theta)$ 5

Since $\tan\theta = \frac{y}{x}$

$\sec^2\theta\, \theta' = \frac{xy' - yx'}{x^2}$

$= \frac{x(-x+\cancel{y}-\cancel{y(3x^2+y^2)}) - y(x+\cancel{y}-\cancel{x(3x^2+y^2)})}{x^2}$

$= -\frac{r^2}{x^2} = -\frac{1}{\cos^2\theta} \Rightarrow \theta' = -1$

   (b) Please show that this system has a stable limit cycle. Please explain your answer.

1 Notice that $(x,y)=(0,0)$ is the only equilibrium point.

1 Also notice that for $r = 1/2$, $r' = \frac{1}{2} - \frac{1}{8}(2\cos^2\theta + 1) \geq \frac{1}{8} > 0 \ \forall\, \theta$,

1 while for $r = 2$, $r' = 2 - 8(2\cos^2\theta + 1) \leq -6 < 0 \ \forall\, \theta$

So $A := \{(x,y) \in \mathbb{R}^2 \mid \frac{1}{2} \leq r \leq 2\}$ is 1 an invariant annulus, and thus by the Poincaré-Bendixson theorem 1, A must contain a stable limit cycle.

2. (10 points) For $f : \mathbb{R}^{n+1} \to \mathbb{R}^n$ and $x_0 \in \mathbb{R}^n$, consider the initial value problem (IVP)

$$\begin{aligned} x' &= f(t, x) \\ x(0) &= x_0 \end{aligned}$$

   (a) Please give the equivalent integral equation for this IVP, then give the expression for $x_k(t)$, the k-th Picard iterate.

2 Integral Equation: $x(t) = x_0 + \int_0^t f(s, x(s))\,ds$

2 Picard Iterate: $x_k = x_0 + \int_0^t f(s, x_{k-1})\,ds$

   (b) Roughly speaking, what must one assume about f for the Picard iterates to converge to a unique solution, and what must one prove about the Picard iterates in order to apply the Arzela-Ascoli theorem.

2 f must be Lipschitz continuous with respect to x for there to be a unique solution.

One must show that $\{x_k\}$ is uniformly bounded 2 and equicontinuous 2 in order to apply the Arzela-Ascoli theorem.