MA3831 Name: Solutions

Final **C Term, 2002**

Show all work needed to reach your answers. You may use any theorem proven in our text, but cite any theorem by page number that you use.

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1. (20 points) Each of the following statements are "intuitively obvious"; please indicate which ones are also false (cross off the false statements).
   (a) If a set is not open, it is closed.
   (b) If S is any set, then $\emptyset \subset S$.
   (c) Since $\mathbb{Z} \subsetneq \mathbb{Q}$, there are a greater number of rationals than integers.
   (d) For $i := \sqrt{-1} \in \mathbb{C}$, $i > 0$.
   (e) If f is a function such that $f : x \mapsto y$, then $f^{-1} : y \mapsto x$. In other words, $y = f(x)$ iff $x = f^{-1}(y)$.

2. (20 points) Please prove that the function $f : \mathbb{R} \to \mathbb{R}$ defined as

$$f(x) = \begin{cases} x\sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

is continuous at $x = 0$. **Hint:** This the same function considered in # 1 (b), p. 90, but here you are asked for a proof. Consider the definition of continuity. You may assume the standard properties of the sine function.

What is needed

1 Given $\epsilon > 0$, one needs $|f(x) - f(0)| = |x\sin(\frac{1}{x}) - 0| = |x||\sin(\frac{1}{x})| < \epsilon$.

Key property 2 Since $|\sin(\frac{1}{x})| \leq 1$ +5 $\forall x$, 3 $|f(x) - f(0)| < \epsilon$ when $|x| < \epsilon$ +6. So one can define $\delta := \epsilon$. Then $|x - 0| = |x| < \delta$ implies $|f(x) - f(0)| < \epsilon$.

4 +5 Definition of Continuity. 5 +4

3. (20 points) Assume that $\{s_n\}$ and $\{t_n\}$ are sequences of reals, that $\{s_n\}$ is bounded, and that $t_n \to 0$ as n increases. Please prove that $s_n t_n \to 0$ as n increases. Note that the definition of convergence is at the top of p. 45 in our text.

+6 1 Since $\{s_n\}$ is bounded, $|s_n| \leq M$ $\forall n \in \mathbb{Z}^+$. 2 +6 Since $t_n \to 0$, given any $\epsilon > 0$, $\exists N$ s.t. if $n > N$, $|t_n| < \epsilon/M$. 3 +1 So for $n > N$, 4 $|s_n t_n| \leq M|t_n| < \epsilon$. +7

So $s_n t_n \to 0$ also.

4. (20 points) Recall that the definition of the *boundary* (p. 62) is $\partial S := \overline{S} \cap \overline{CS}$. Please show that if p is a point on the boundary, than $\forall\ \epsilon > 0$, $B_\epsilon(p)$ has a nonempty intersection with both S and CS.

Let $p \in \partial S = \overline{S} \cap \overline{CS}$. Then $p \in \overline{S}$ and $p \in \overline{CS}$. Suppose $p \in S$; then $p \notin CS$. Suppose that $\exists\ \epsilon > 0$ s.t. $B_\epsilon(p) \cap CS = \emptyset$. Then since $B_\epsilon(p)$ is open, $C(B_\epsilon(p))$ is closed, and $CS \subset C(B_\epsilon(p))$. So $\overline{CS} \subset C(B_\epsilon(p))$ and this contradicts the fact that $p \in \overline{CS}$. So $\forall \epsilon > 0$, $B_\epsilon(p) \cap CS \neq \emptyset$. Since the same argument works if one starts with $p \in CS$, the result follows.

5. (20 points) Let (E, d) be a metric space. Suppose that $f(x, y) := 2d(x, y)$ and $g(x, y) := (d(x, y))^2$. Please prove or disprove each of the following statements:

(a) (E, f) is a metric space.

+1 +1 Mention at least once.

(1) Since $d(p,q) \geq 0\ \forall p,q \in E$, then $f(p,q) = 2d(p,q) \geq 0\ \forall p,q \in E$ $(2 > 0!)$. +1

(2) Since +1 $d(p,q) = 0$ iff $p = q$, then $f(p,q) = 2d(p,q) = 0$ iff $p = q$ +1 $(2 \neq 0!)$.

(3) Since $d(p,q) = d(q,p)\ \forall p,q \in E$, then $f(p,q) = 2d(p,q) = 2d(q,p) = f(q,p)\ \forall p,q \in E$ +2

(4) Since +1 $d(p,q) \leq d(p,r) + d(r,q)$ +2 $\forall p,q,r \in E$, then $\forall p,q,r \in E$

$f(p,q) = 2d(p,q) \leq 2\left(d(p,r) + d(r,q)\right) = 2d(p,r) + 2d(r,q) \leq f(p,r) + f(r,q)$,

+1 +1 +1

Since f satisfies all 4 of the defining properties, it is a metric.

(b) (E, g) is a metric space.

Suppose there are 3 points s.t. $d(p,q) = 1$, $d(p,r) = \frac{1}{2}$ and $d(r,q) = 1/2$. For example, let $p,q,r \in \mathbb{R}$ with $p = 0$, $q = 1$, $r = 1/2$, and use the Euclidean metric. Then $g(p,q) = (d(p,q))^2 = 1 > g(p,r) + g(r,q)$

$$= (d(p,r))^2 + (d(r,q))^2 = \left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{2}\right)^2 = 1/2.$$

So this g can not satisfy the triangle inequality, hence it is no metric.

Any valid counter example +10