MA3831 Name: Solutions

Final C Term, 2004

Show all work needed to reach your answers. You may use any theorem proven in our text or on the compactness sheet, but cite by page number or name any theorem that you use.

1. (30 points) Suppose that (E, d) and (E', d') are metric spaces. Let a function $f : E \to E'$ be *continuous* on E. Please make three distinct statements about f that are equivalent to "continuous".

(a) $\forall p_0 \in E$ (+1), given $\epsilon > 0$ (+2), $\exists \delta > 0$ (+2) s.t. $d(f(p), f(p_0)) < \epsilon$ (+3) whenever $d(p, p_0) < \delta$ (+2).

(b) $\forall p_0 \in E$ (+1), given $\epsilon > 0$ (+2), $\exists \delta > 0$ (+2) s.t. $f(B_\delta(p_0))$ (+2) $\subset$ (+1) $B_\epsilon(f(p_0))$ (+2).

(c) If $G \subset E'$ is open, then $f^{-1}(G) \subset E$ is also open (the inverse image of any open set is open)

2. (20 points)

(a) Please explain precisely why $\mathbb{Q}$ is incomplete.

A set (or space) is complete if every Cauchy (+2) sequence converges to a point in the set (or space). But we know that there are many Cauchy sequences of rationals that do not converge to rationals, i.e.,

$\{3, \frac{31}{10}, \frac{314}{100}, \frac{3141}{1000}, \frac{31415}{10000}, \frac{314159}{100000}, \ldots\}$ (+2)

converges to $\pi \notin \mathbb{Q}$.

(b) Please explain precisely why $\mathbb{C}$ is not ordered.

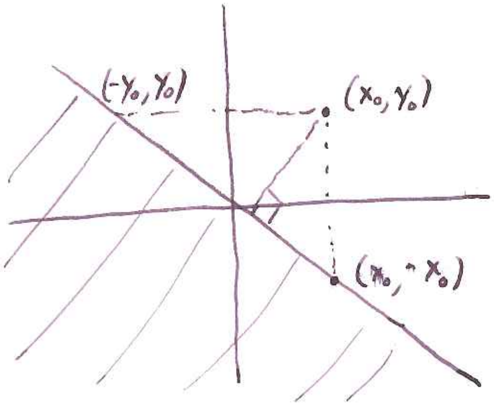
Suppose that $\mathbb{C}$ is ordered: then either $i > 0$ or $i < 0$. Without lose of generality, assume $i > 0 \iff i \in \mathbb{C}^+$. Then $i \cdot i = -1 > 0$ (the product of two positives is positive). This seems problematic, but to get a clear contradiction, multiply again by $i \in \mathbb{C}^+$: $i \cdot (i \cdot i) = -i > 0$. So $i \in \mathbb{C}^+$ and $-i \in \mathbb{C}^+ \rightarrow\leftarrow$. Thus $\mathbb{C}$ can not be ~~orderdd~~.

ordered.

+4

S =

3. (25 points) Please show that the set $\{ (x,y) \in \mathbb{R}^2 \mid y \le -x \}$ is closed.
Hint: Construct an appropriate open ball.



To show that S is closed, it is sufficient to show that $\complement S$ is open. +5 $\complement S = \{(x,y) \in \mathbb{R}^2 \mid y > -x\}$ +5

Let $(x_0, y_0) \in \complement S$. Then the closest point on the line $y = -x$ is the midpoint $\left(\frac{x_0 - y_0}{2}, \frac{y_0 - x_0}{2}\right)$, and the +4 distance

$$d\left((x_0,y_0), \left(\tfrac{x_0-y_0}{2}, \tfrac{y_0-x_0}{2}\right)\right) = \sqrt{\left(x_0 - \tfrac{x_0-y_0}{2}\right)^2 + \left(y_0 - \tfrac{y_0-x_0}{2}\right)^2}$$

+5 $= \sqrt{2\left(\frac{x_0+y_0}{2}\right)^2} = \frac{x_0+y_0}{\sqrt{2}}$. Let $\epsilon := \frac{x_0+y_0}{2}$ +2

So $B_\epsilon(x_0, y_0) \subset \complement S \Rightarrow \complement S$ is open $\Rightarrow S$ is closed.

+4

4. (25 points) Let (E, d) be a metric space, and let F_1 and F_2 be nonempty, disjoint compact sets. Please show that there exists $\epsilon > 0$ such that $d(p,q) > \epsilon$ for all $p \in F_1$ and $q \in F_2$.
Hint: Give a proof-by-contradiction; explain why there exists sequences of points from each set, $\{p_n\} \subset F_1$ and $\{q_n\} \subset F_2$, such that $d(p_n, q_n) < 1/n \ \forall\, n \in \mathbb{Z}^+$.

(Contradiction) Suppose that $\forall \epsilon > 0\ \exists\, p \in F_1$ and $q \in F_2$ s.t. $d(p,q) < \epsilon$. Then $\exists\, p_n \in F_1$ and $q_n \in F_2$ s.t. $d(p_n, q_n) < 1/n\ \forall n \in \mathbb{Z}^+$. Since F_1 is compact, the sequence $\{p_n\} \subset F_1$ contains a subsequence $\{p_{n_k}\}$ s.t. $\exists\, p^* \in F_1$ with $p_{n_k} \to p^*$ (Corollary 1, either from the Compactness Theorems Sheet, or from p. 56 in Rosenlicht). Also by the Theorem of p. 47, $p^* \in F_1$.

Now given any $\delta > 0$, $\exists\, N \in \mathbb{Z}^+$ s.t. for $n_k > N$, $d(p_{n_k}, p) < \delta/2$ (the definition of convergence) and $d(p_{n_k}, q_{n_k}) < \frac{1}{n_k} < \delta/2$ (the initial assumption). So

$$d(q_{n_k}, p^*) \le d(q_{n_k}, p_{n_k}) + d(p_{n_k}, p^*) < \delta$$

Hence by definition $q_{n_k} \to p^*$, and again by the Theorem on p. 47, $p^* \in F_2$. But then $p^* \in F_1 \cap F_2$, meaning that F_1 and F_2 are not disjoint. $(\rightarrow\leftarrow)$