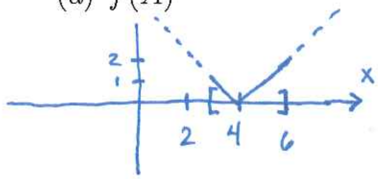
MA3831 **Name:** *Solutions*

Midterm, Part 1 Closed Book **C Term, 2002**

(1) (24 points) If $f : \mathbb{R} \to \mathbb{R}$ is given by $f : x \mapsto |x - 4|$, and $A = \{t \in \mathbb{R} \,|\, 3 \le t \le 6\}$, $B = \{t \in \mathbb{R} \,|\, t < 5\}$ and $C = \{-1, 0, 1\}$. Please find each of the following:

(a) $f(A)$



So $f : A \to f(A) = \{y \in \mathbb{R} \,/\, 0 \le y \le 2\}$

Notice that one can not simply work from the endpoints of the intervals.

(b) $f^{-1}(B) = \{x \in \mathbb{R} \,/\, |x-4| < 5\}$

since $|x-4| < 5 \iff \left(x - 4 < 5 \;\vee\; -x+4 = -(x-4) < 5\right)$

$\iff -5 < x - 4 < 5 \iff -1 < x < 9$

(c) $f(f^{-1}(C))$

$f^{-1}(C) = \{4, 3, 5\}$ because $f^{-1}(-1) = \emptyset$, $f^{-1}(0) = \{4\}$ & $f^{-1}(1) = \{3, 5\}$

So $f(f^{-1}(C)) = \{0, 1\}$

(2) (15 points) Please solve the following for x:

$$\frac{-3}{x-4} > x$$

- First notice that $x \neq 4$.
- For $x > 4$, $-3 > x(x-4) = x^2 - 4x \iff x^2 - 4x + 3 = (x-3)(x-1) < 0$
  For the product of two factors to be negative, one ~~factor~~ must be positive and one must be negative. This requires that $1 < x < 3$. But since $x > 4$, no x satisfy these requirements.
- For $x < 4$, $-3 < x(x-4) = x^2 - 4x \iff 0 < x^2 - 4x + 3 = (x-3)(x-1)$
  Now one needs both factors to be positive, or both to be negative. So either $x < 1$ or $x > 3$, and therefore the set of points satisfying these requirements is

$$\{x \in \mathbb{R} \,/\, x < 1 \;\vee\; 3 < x < 4\}$$

There's no way to do this one without cases.

(3) (15 points) What is the greatest lower bound (glb) and least upper bound (lub) of the following set? Please carefully explain your answer and give it exactly (no approximations).

$$\{\pi, \pi - \frac{1}{2}, \pi - \frac{2}{3}, \pi - \frac{3}{4}, \dots\}$$

Since the sequence $\pi, \pi - 1/2, \pi - \frac{2}{3}, \dots, \pi - \frac{n-1}{n}, \dots$ is decreasing, the lub is the first member: $\underline{lub = \pi}$.

Since $\lim_{n\to\infty} \frac{n-1}{n} = \lim_{n\to\infty} 1 - 1/n = 1$, the glb is $\underline{\pi - 1}$.

(4) (6 points) Please write the following statement in *if-then* form: "**C** is a necessary condition for **B**."

This statement is equivalent to "$B \Rightarrow C$" or in if-then form "If B, then C."

These are counterpositive to "$\neg C \Rightarrow \neg B$" or "without C, there can not be B."

(5) (20 points) Please prove the following: If $f : X \to Y$, f is one-to-one, and $A, B \subset X$, then $f(A) \cap f(B) \subset f(A \cap B)$.

Let $y \in f(A) \cap f(B)$. Then $y \in f(A)$ and $y \in f(B)$. So $\exists x_1 \in A$ and $x_2 \in B$ such that $f(x_1) = y$ and $f(x_2) = y$. But since f is one-to-one, $f(x_1) = f(x_2) \Rightarrow x_1 = x_2 =: x$. And $x \in A, x \in B \Rightarrow x \in A \cap B$. So $y = f(x) \in f(A \cap B)$. Thus $f(A) \cap f(B) \subset f(A \cap B)$.

If f is not one-to-one, then x_1 and x_2 may be distinct, and so $A \cap B = \emptyset$.