MA3831 Name: Solutions

Midterm, Part 1 $\boxed{\text{Closed Book}}$ C Term, 2004

1. (20 points) If A and B are sets, please show that $\mathcal{C}A \cup \mathcal{C}B \subset \mathcal{C}(A \cap B)$.

Let $x \in \mathcal{C}A \cup \mathcal{C}B$. Then $x \in \mathcal{C}A$ or $x \in \mathcal{C}B$. So $x \notin A$ or $x \notin B$, and thus $x \notin A \cap B$. $\therefore$ $x \in \mathcal{C}(A \cap B)$, and $\mathcal{C}A \cup \mathcal{C}B \subset \mathcal{C}(A \cap B)$.
QED

2. (15 points) For each of the following inequalities, please determine which $x \in \mathbb{R}$ satisfy the inequality.

(a)

$$| x^2 - 4 | > -3x$$

Case 1: $x^2 - 4 \geq 0 \Leftrightarrow (x-2)(x+2) \geq 0 \Leftrightarrow x \leq -2$ ~~or~~ $x \geq 2$.
In this case, $x^2 - 4 > -3x \Leftrightarrow x^2 + 3x - 4 > 0 \Leftrightarrow (x+4)(x-1) > 0$
So $x < -4$ or $x > 1$. Combining these conditions, we have $\{x \in \mathbb{R} \mid x < -4 \vee x \geq 2\}$

Case 2: $x^2 - 4 \leq 0 \Leftrightarrow (x-2)(x+2) \leq 0 \Leftrightarrow -2 \leq x \leq 2$.
For this case, $-x^2 + 4 > -3x \Leftrightarrow 0 > x^2 - 3x - 4 \Leftrightarrow 0 > (x-4)(x+1)$
So $-1 < x < 4$. Again combining the conditions, we have $\{x \in \mathbb{R} \mid -1 < x \leq 2\}$.

$\{x \in \mathbb{R} \mid x < -4 \vee -1 < x\} = \{x \in \mathbb{R} \mid x < -4\} \cup \{x \in \mathbb{R} \mid -1 < x\}$

(b)

$$\frac{-7}{x+3} \leq 0$$

Because $x+3$ is in the denominator, $x + 3 \neq 0$. If $x > -3$, then $x+3 > 0$, and since $-7 \leq 0$, $\{x \in \mathbb{R} \mid x > -3\}$ satisfies this inequality. If $x < -3$, then $x + 3 < 0$. But if $x + 3 < 0$, then $-7/_{x+3} > 0$, so none of these values satisfy the inequality.

$\{x \in \mathbb{R} \mid x > -3\}$

3. (15 points) What is the greatest lower bound (glb) and least upper bound (lub) of the following set? Please carefully explain your answer and give it exactly (no approximations).

$$\{\pi,\ \pi+\frac{1}{2},\ \pi-\frac{2}{3},\ \pi+\frac{3}{4},\ \pi-\frac{4}{5},\ \pi+\frac{5}{6},\ \pi-\frac{6}{7},\ \ldots\}$$

This set contains two sequences of numbers, one increasing of the form $\pi+\frac{1}{2}, \pi+\frac{3}{4}, \pi+\frac{5}{6}, \ldots$ and the other decreasing of the form $\pi-\frac{2}{3}, \pi-\frac{4}{5}, \pi-\frac{6}{7}, \ldots$

Since $\lim_{n\to\infty}(\pi+\frac{n-1}{n}) = \pi + \lim_{n\to\infty}(1-\frac{1}{n}) = \pi+1$, and since $\lim_{n\to\infty}(\pi-\frac{n-1}{n}) = \pi-1$,

lub $= \pi+1$ and glb $= \pi-1$.

4. (20 points) Please prove or disprove: If $x, y \in \mathbb{Q}$, then $x+y \in \mathbb{Q}$.

Proposition: If $x, y \in \mathbb{Q}$, then $x+y \in \mathbb{Q}$.

Proof: If $x, y \in \mathbb{Q}$, then $\exists\, n, m \in \mathbb{Z}$ ($n \in \mathbb{Z}, m \in \mathbb{Z}^+$) such that $x = \frac{n}{m}$.

Also $\exists\, p, q \in \mathbb{Z}$ ($p \in \mathbb{Z}, q \in \mathbb{Z}^+$) s.t. $y = p/q$. So

$$x+y = \frac{n}{m} + \frac{p}{q} = \frac{n\cdot q}{m\cdot q} + \frac{m\cdot p}{m\cdot q} = \frac{nq+mp}{mq}$$

Since the integers are closed under multiplication and addition, $nq+mp,\ mq \in \mathbb{Z}$.

Since $x+y$ is the ratio of two integers with $mq \in \mathbb{Z}^+$, $x+y \in \mathbb{Q}$.

MA3831 Name: Solutions

Midterm, Part 2 Open Book **C Term, 2004**

1. (15 points) Please prove in detail that if $\mathbb{F}$ is any field with $a \in \mathbb{F}$, and $a + a = a$, then $a = 0$. Please cite one of the five field properties (roman numerals, p. 16) to justify each step. || Notice that one can avoid introducing subtraction.

$$a \overset{IV}{=} a + 0 \overset{V}{=} a + (a + (-a)) \overset{III}{=} (a+a) + (-a) = a + (-a) \overset{V}{=} 0$$

2. (15 points) Let $X \subset \mathbb{R}$, $X \neq \emptyset$, and assume that X is bounded above. By the least upper bound property of the reals, $\exists\ a := \underline{l.u.b.}\{X\} \in \mathbb{R}$. ~~Suppose that~~ Finally, let $a \notin X$. Please show that X is infinite (perhaps countable, perhaps uncountable, but not finite).

Consider the sequence $\{a-1, a-1/2, a-1/3, a-1/4, \ldots, a-\frac{1}{n}, \ldots\}$
Since a is the <u>least</u> upper bound of X, $\exists\ x_1 \in X$ s.t. $a-1 < x_1 < a$. By LUB 5, $\exists\ n_2 \in \mathbb{Z}^+$ s.t. $x_1 < a - \frac{1}{n_2} < a$. But again, since a is the <u>least</u> upper bound of X, $\exists\ x_2 \in X$ s.t. $a - \frac{1}{n_2} < x_2 < a$. Continuing this process, one can construct a countable sequence $\{x_n\} = \{x_1, x_2, x_3, \ldots, x_n, \ldots\}$ s.t. $x_n \in X\ \forall n$; $x_n \nearrow a$; and $a \neq x_n$ for any n. So X must be infinite.

<u>Alternative Proof</u> (contradiction):

Suppose X is finite. Then $\exists\ n \in \mathbb{Z}^+$ s.t. $X = \{x_1, x_2, \ldots, x_n\}$. Let $x := \max_i\{x_i\}$ (This value can always be found by making n comparisons). Since $a \notin X$, but a is an upper bound $x < a$. But also $x_i \leq x\ \forall i$, $1 \leq i \leq n$. So x is an upper bound for X less than a. $\rightarrow\leftarrow$

So X must not be finite.