MA3831 Name: Solutions Section: ______

Final (Open Book, Open Notes) A Term, 2011

If you are/were a WPI undergrad, what was your first math course at WPI? ______

This is an open book/open notes exam. Show all work needed to reach your answers. You may use any theorem proven in our text or on the compactness sheet, but cite by page number or name any theorem that you use.

5 points each

1. (20 points) Suppose that (E, d) is a metric space, and let $G \subset E$ be *open*. Please make four distinct statements about G that are equivalent to "open".

(a) For any $p \in G$, $\exists\ \epsilon > 0$ s.t. $B_\epsilon(p) \subset G$.

(b) $\mathcal{C}G$ is closed.

(c) $G = G^\circ$ (interior of G)

(d) $\partial G \cap G = \emptyset$ (G and its boundary have no points in common)

2. (20 points) For $S \subset \mathbb{R}^n$, suppose that S is bounded. Further suppose that $F \subset S$ and that F is closed. Please show that F is compact.

Since S is bounded (+2), $\exists\ M > 0$ s.t. $S \subset B_M(0)$ (+4). So also $F \subset B_M(0)$ (+2) $\Rightarrow$ F is both closed and bounded in $\mathbb{R}^n$ (+4). By the Heine-Borel theorem (+4), closed and bounded in $\mathbb{R}^n \Rightarrow$ compact (+4).

3. (20 points) Please explain precisely why $\mathbb{Q}$ is incomplete.

+5

A set (or space) is complete if every Cauchy sequence is convergent to a point in the set +5 (or space). But there are many sequences in $\mathbb{Q}$ that converge to points in $\mathbb{R}-\mathbb{Q}$, e.g.

$$\left\{1, \frac{14}{10}, \frac{141}{100}, \frac{1414}{1000}, \frac{14142}{10000}, \ldots\right\}$$ +5

converges to $\sqrt{2} \notin \mathbb{Q}$.

+5

4. (25 points) Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined as

$$f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^2} & (x,y) \neq (0,0) \\ \\ 0 & (x,y) = (0,0) \end{cases}$$

Please discuss as completely as possible the continuity of this function at all points $(x,y) \in \mathbb{R}^2$. If the function f is continuous at a point, prove that f is continuous. If f is discontinuous at a point, please explain why f is discontinuous. **Hint**: Consider separately two cases: (1) when $(x,y) = (0,0)$ and (2) when $(x,y) \neq (0,0)$.

Case (1): At $(x,y)=(0,0)$, f is continuous because in polar coordinates, +5

$$f(r\cos\theta, r\sin\theta) = \begin{cases} r\cos\theta\sin^2\theta & r>0 \\ 0 & r=0 \end{cases}$$

+5

So $|f(x,y)-f(0,0)| = |r\cos\theta\sin^2\theta - 0| < r \quad \forall (x,y)$

+5

Since $d((x,y),(0,0)) = r$, one can take $\delta = \varepsilon$.

+5

Case (2): For $(x,y) \neq (0,0)$, this is a rational function whose denominator is not zero. So by the basic proposition for the sum, product and quotient of continuous functions (p. 75), this f is continuous away from the origin. +5

5. (15 points) Assume that $A \subset \mathbb{R}$ is non-empty. Furthermore assume that every increasing and bounded sequence in A is convergent in A. Please give a counterexample to disprove the statement "A is compact", and explain why your counterexample works.

There are examples that are either not closed or are not bounded (or both not closed and not bounded):

Not Closed: Let $A = (0,1]$. Suppose $\{a_k\} \subset A$ is increasing; since the sequence is bounded above by 1, by the LUB property of $\mathbb{R}$, $\exists\, a = \mathrm{lub}(\{a_k\}) \le 1$ $\Rightarrow a \in A$. But $\{\frac{1}{n}\} \subset A$ and yet $\{\frac{1}{n}\} \searrow 0 \notin A$.

Not Bounded: Let $A = [0, +\infty)$ (or even $A = \mathbb{R}$). If $\{a_k\} \subset A$ is increasing and bounded above, then by the LUB property of $\mathbb{R}$, $\exists\, a = \mathrm{lub}(\{a_k\}) \in A$ s.t. $a_k \nearrow a$.