MA3831 Name: Solutions

Final **C Term, 2003**

Show all work needed to reach your answers. You may use any theorem proven in our text or on the compactness sheet, but cite by page number or name any theorem that you use.

1. (30 points) Let a set $S \subset \mathbb{R}$ be *compact.* Please make three distinct statements about S that are equivalent to "compact". **Hint**: The first two are things we talked about a lot in class; the third you need to think about yourself. For the third, you may want to consider a sequence in S.

   (a) Every open cover of S has a finite subcover. +10

   (b) S is closed and bounded. +10

   (c) Every sequence in S has a convergent subsequence. +10

2. (20 points) For each of the following sets $S \subset \mathbb{R}$, please give $\bar{S}$ (the closure of S), S^o (the interior of S), and ∂S (the boundary of S).

   (a) $S = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1, y > 0\}$

   $\bar{S} =$ $S \cup \{(x,y) \mid y=0, -1 \leq x \leq 1\}$ +2 $\quad S^o =$ $\{(x,y) \mid x^2+y^2<1, y>0\}$ +2 $\quad \partial S =$ $\{(x,y) \mid x^2+y^2=1, y>0\} \cup \{(x,y) \mid y=0, -1\leq x\leq 1\}$ +2

   Notice that in each case $\bar{S} = S^o \cup \partial S$

   (b) $S = \{x \in \mathbb{R} \mid x = 1/n \text{ for } n \in \mathbb{Z}^+\}$

   $\bar{S} =$ $S \cup \{0\}$ +2 $\quad S^o =$ $\emptyset$ +2 $\quad \partial S =$ $S \cup \{0\}$ +2

   (c) $S = \mathbb{Q}$

   $\bar{S} =$ $\mathbb{R}$ +2 $\quad S^o =$ $\emptyset$ +2 $\quad \partial S =$ $\mathbb{R}$ +2

   (d) If $\bar{S} = S^o$, what can be said about S?

   Then either $S = \mathbb{R}$ or $S = \emptyset$ (the only two sets that are both open and closed). +2

3. (25 points) Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined as

$$f(x,y) = \begin{cases} \dfrac{x^2 - y^2 - x + y}{x - y} & x \neq y \\[2ex] a & x = y \end{cases}$$

where a is a constant. Considering the definition of continuity, please discuss as completely as possible the continuity of this function at all points $(x,y) \in \mathbb{R}^2$. For what (if any) value of a is f continuous at $(0,0)$?

Because f is a rational function, +5 it is continuous +5 except when the denominator is zero (i.e., for $x \neq y$). For points on the line $y = x$, consider values of (x,y) near this line. Here since $x \neq y$,

$$f(x,y) = \frac{x^2 - y^2 - x + y}{x - y} = \frac{(x-y)(x+y) - (x-y)}{x-y} = \frac{(x+y) - 1}{1} \; \text{+4} = x + y - 1$$

Since $\lim_{x \to y} f(x,y) = \lim_{x \to y} (x + y - 1) = 2y - 1$, f will be continuous at (y,y) when +3 $f(y,y) = a = 2y - 1 \Rightarrow y = \frac{a+1}{2}$. Thus f can not be continuous for $y = x$ except at one point: +4 $\left(\frac{a+1}{2}, \frac{a+1}{2}\right)$. So for f to be continuous at $(0,0)$ we would need $a = -1$.

+4

4. (25 points) Consider the sequence $\{p_n\} = \{p_1, p_2, p_3, \ldots, p_n, \ldots\}$. Suppose that this sequence is Cauchy and has a convergent subsequence $\{p_{n_k}\}$ which converges to a point p. Please prove or disprove: the sequence $\{p_n\}$ converges to p.

Proposition: $p_n \to p$.

Proof: Given $\epsilon > 0$, $\exists N_1 \in \mathbb{Z}^+$ s.t. +6 $\forall n_k > N_1$, $d(p, p_{n_k}) < \epsilon/2$.

Also $\exists N_2 \in \mathbb{Z}^+$ s.t. $\forall n_k > N_2$, +6 $d(p_{n_k}, p_n) < \epsilon/2$. Let

+3 $N := \max\{N_1, N_2\}$. Then by the +1 triangle inequality, $\forall n > N$, $\exists n_k > N$ s.t.

+4

$$d(p, p_n) \leq d(p, p_{n_k}) + d(p_{n_k}, p_n) < \epsilon/2 + \epsilon/2 = \epsilon$$

Thus $p_n \to p$ by definition. QED

+5