MA3831 Name: Solutions Section: ______

Midterm (closed book, closed notes) A Term, 2011

If you are/were a WPI undergrad, what was your first math course at WPI? ______

Show all work needed to reach your answers. You may use any theorem we discussed in class, but cite by name any theorem you use.

1. (20 points) If A and B are sets, please show that $\mathcal{C}(\mathcal{A} \cup \mathcal{B}) \subset \mathcal{C}\mathcal{A} \cap \mathcal{C}\mathcal{B}$ (the other containment direction is also true, but not part of this question).

Suppose that +4 $x \in C(A \cup B)$. Then +4 $x \notin A \cup B$, and thus $x \notin A$ +4 and $x \notin B$. So $x \in CA$ and +4 $x \in CB \Rightarrow x \in$ +4 $CA \cap CB$. Hence $C(A \cup B) \subset CA \cap CB$.

2. (20 points) If $x_1, x_2 \in \mathbb{Q}$, show in detail that their product is rational: $x_1 x_2 \in \mathbb{Q}$.

Suppose that +4 $x_1, x_2 \in \mathbb{Q}$. Then $\exists$ $p_1, p_2 \in \mathbb{Z}$ and +4 $q_1, q_2 \in \mathbb{Z}^+$ such that $x_1 = \frac{p_1}{q_1}$ and $x_2 = p_2/q_2$. So using the field properties, one finds that

+4

$$x_1 \cdot x_2 = \left(\frac{p_1}{q_1}\right)\cdot\left(\frac{p_2}{q_2}\right) = \left(p_1 \cdot \frac{1}{q_1}\right)\cdot\left(p_2 \cdot \frac{1}{q_2}\right) = p_1 \cdot p_2 \cdot \frac{1}{q_1} \cdot \frac{1}{q_2} = \frac{p_1 p_2}{q_1 q_2}$$

Since both $\mathbb{Z}$ and $\mathbb{Z}^+$ are closed under +4 multiplication, $p_1 p_2 \in \mathbb{Z}$ and $q_1 q_2 \in \mathbb{Z}^+$. Thus $x_1 x_2$ is indeed the +4 quotient of integers and therefore rational.

3. (25 points) For which $x \in \mathbb{R}$ is $x + 2 < \frac{1}{|x|}$? Please show all needed steps, and give your final answer as the union of intervals.

(+1) Because of the $|x|$ in the denominator, $x \neq 0$.

Case 1: (+1) For $x > 0$, $x + 2 < \frac{1}{x}$ (+1) $\Leftrightarrow$ $x^2 + 2x - 1 < 0$. (+1)

Now by the quadratic formula, (+1) the roots of $x^2 + 2x - 1 = 0$ are

$x_{\pm} = \frac{-2 \pm \sqrt{4+4}}{2} = -1 \pm \sqrt{2}$ (+2). Thus $(x + 1 - \sqrt{2})(x + 1 + \sqrt{2}) < 0$, (+2) and this inequality implies $-1 - \sqrt{2} < x < -1 + \sqrt{2}$. (+2) But for this case, $x > 0$ so (+1) $0 < x < -1 + \sqrt{2}$.

Case 2: (+1) For $x < 0$, $x + 2 < \frac{1}{-x}$ (+1) $\Leftrightarrow$ $-x^2 - 2x < 1$ (+2) (recall that $-x > 0$). (+1)

So $x^2 + 2x + 1 > 0$, (+1) and therefore $(x+1)^2 > 0$. (+2) This inequality is always satisfies, except at $x = -1$. (+2) Hence in this case, $-\infty < x < -1$ (+1) or $-1 < x < 0$.

Combining these cases

Solution: $(-\infty, -1) \cup (-1, 0) \cup (0, -1 + \sqrt{2})$ (+2)

3 points each

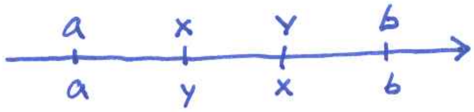
4. (15 points) **Please fill in the blanks.**

Suppose $f : X \to Y$ is a function. Then $f(x)$ is defined for all ($\forall$) $x \in X$, and f maps X onto $f(X)$. Also if $y \in \text{Range}(f)$, then there exists $x \in X$ such that $y = f(x)$. And if $A \subset X$, then $f^{-1}(f(A)) \supseteq A$. Finally f^{-1} is itself a function iff f is one-to-one.

5. (10 points) Please show that if a, b, x, $y \in \mathbb{R}$, and $a < x < b$, $a < y < b$, then $|y - x| < b - a$.

Since $a < x$, $-x < -a$. Since $y < b$, one can add the inequalities: $y - x < b - a$. Similarly $x < b \Rightarrow -b < -x$, and $-b < -x$ combines with $a < y$ to yield $-(b-a) < y - x$. Thus $-(b-a) < y - x < b - a$ which implies that

$$|y - x| < b - a$$



There were a variety of answers to this one, many of them correct or basically correct. The key point was to give an exact proof!

6. (10 points) Suppose $S \subset \mathbb{R}$ is a nonempty set that is bounded above. Can S have more than one least upper bound? Please explain in detail why or why not.

The lub(s) is unique. If it was not, then let a_1 and a_2 both be least upper bounds for S. Then both a_1 and a_2 must be bounds for S, and since a_1 is a lub(s), then $a_1 \leq a_2$. But the other direction also works: a_1 is a bound and a_2 is a lub(s), so $a_2 \leq a_1$. All of this implies that $a_1 = a_2$ and thus there is indeed only one lub(s).

Again, there were a variety of answers, but the correct ones all henged on a_1 and a_2 both being bounds and lub(s). Full credit required