MA3831 **Name:** Solutions

Midterm, Part 1 Closed Book **C Term, 2003**

(1) (20 points) For which $x \in \mathbb{R}$ is $3x^2 - 2 < |x|$? Please justify your answer.

For $x \geq 0$, $3x^2 - 2 < x \Rightarrow 3x^2 - x - 2 < 0 \Rightarrow (3x+2)(x-1) < 0$

For a product to be negative, one factor must be positive and the other negative. In this case, that means $-2/3 < x < 1$. But we already have the $x \geq 0$, so the range is $0 \leq x < 1$.

For $x \leq 0$, $3x^2 - 2 < -x \Rightarrow 3x^2 + x - 2 < 0 \Rightarrow (3x-2)(x+1) < 0$

So here $-1 < x < 2/3$. But in this case $x \leq 0$, so the range in this case is $-1 < x \leq 0$.

Combining these two cases, one finds that $\underline{-1 < x < 1}$.

(2) (10 points) Please give the contrapositive of the following statement: "If the sky is falling, then chicken Little was right!"

"If chicken Little was not right, then the sky is not falling!"

The contrapositive of "$A \Rightarrow B$" is "$\neg B \Rightarrow \neg A$".

(3) (20 points) Let $f : \mathbb{R} \to S$ be given by $f(x) = 3x^2 + 7$. Using the definition of *onto*, please explain why f is onto the set $S = \{y \in \mathbb{R} \,|\, y \geq 7\}$.

Suppose $y \in S$. One must find $x \in \mathbb{R}$ s.t. $f(x) = y$. But then $3x^2 + 7 = y \Rightarrow x^2 = \frac{y-7}{3} \geq 0 \;\; \forall y \in S$. But since $\frac{y-7}{3} \geq 0$, its square root is real. Hence $x = \sqrt{\frac{y-7}{3}} \in \mathbb{R}$, and for this x, $f : x \mapsto y$. QED Note that one could also have chosen $x = -\sqrt{\frac{y-7}{3}}$ if one wished.

(4) (20 points) What is the greatest lower bound (glb) and least upper bound (lub) of the following set? Please carefully explain your answer and give it exactly (no approximations).

$$S := \left\{ \sqrt{11}, \sqrt{11 + \sqrt{11}}, \sqrt{11 + \sqrt{11 + \sqrt{11}}}, \ldots \right\}$$

Since $\forall x > 0$, $\sqrt{11} < \sqrt{11 + x}$, the elements in this set increase (moving from left to right). Thus $glb(S) = \sqrt{11}$. The set is then either not bounded above, or the sequence converges to some real number. Notice that $x_{n+1} = \sqrt{11 + x_n}$. Let's assume that $x_n \to x$. Then also $x_{n+1} \to x$, and so in this case $x = \sqrt{11 + x} \Rightarrow x^2 = 11 + x \Rightarrow x^2 - x - 11 = 0$. Since this quadratic does not factor, one must use the quadratic formula: $x = \frac{1 + \sqrt{1 + 4(11)}}{2} = \frac{1}{2} + \frac{3\sqrt{5}}{2} > 0$ (the negative root is not a possible choice in this case). So at least if $x_n \to x$, then $lub(S) = \frac{1}{2} + \frac{3\sqrt{5}}{2}$.

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Midterm, Part 2 Open Book **C Term, 2003**

1. (15 points) Please prove in detail the following identity where $a, b \in \mathbb{R}$. Please cite the Property (Roman numerals), Order (O numbers) and Field (F numbers) to justify each step.

$$\frac{b}{a} - \frac{a}{b} = \left(1 - \frac{a}{b}\right)\left(1 + \frac{b}{a}\right)$$

* Definition of minus.

$$\frac{b}{a} - \frac{a}{b} \overset{IV}{=} \left(\frac{b}{a} - \frac{a}{b}\right) + 0 \overset{V}{=} \left(\frac{b}{a} - \frac{a}{b}\right) + (1-1) \overset{*}{=} \left(\frac{b}{a} - \frac{a}{b}\right) + 1 + (-1)$$

$$\overset{I}{=} 1 + \left(\frac{b}{a} - \frac{a}{b}\right) + (-1) \overset{*}{=} 1 + \frac{b}{a} + \left(-\frac{a}{b}\right) + (-1)$$

$$\overset{II}{=} \left(1 + \frac{b}{a}\right) - \left(\frac{a}{b} + 1\right) \overset{V}{=} \left(1 + \frac{b}{a}\right) - \left(\frac{a}{b} + \frac{ab}{ab}\right)$$

$$\overset{F9}{=} \left(1 + \frac{b}{a}\right) - \left(\frac{a}{b} + \frac{a}{b} \cdot \frac{b}{a}\right) \overset{III}{=} \left(1 + \frac{b}{a}\right) - \frac{a}{b}\left(1 + \frac{b}{a}\right)$$

$$\overset{III}{=} \left(1 + \frac{b}{a}\right)\left(1 - \frac{a}{b}\right) \overset{I}{=} \left(1 - \frac{a}{b}\right)\left(1 + \frac{b}{a}\right)$$

This is not the minimum number of steps.

y is irrational

2. (15 points) Please prove or disprove: If $x \in \mathbb{Q}$ and $y \in \mathbb{R} - \mathbb{Q}$, then $x + y$ may be rational or irrational depending on the exact values of x and y.

Suppose $x + y \in \mathbb{Q}$. Then $\exists\, a, b \in \mathbb{Z}$ s.t. $x + y = \frac{a}{b}$. But since $x \in \mathbb{Q}$, $\exists$ $c, d \in \mathbb{Z}$ s.t. $x = c/d$. Thus $y = (x+y) - x = \frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd} \in \mathbb{Q}$ $\rightarrow\leftarrow$

So $x + y \notin \mathbb{Q} \Rightarrow x + y$ is irrational. QED