MA3831 **Name:** ____________________

Final **C Term, 2002**

Show all work needed to reach your answers. You may use any theorem proven in our text, but cite any theorem by page number that you use.

1. (20 points) Each of the following statements are "intuitively obvious"; please indicate which ones are also false (cross off the false statements).
   (a) If a set is not open, it is closed.
   (b) If S is any set, then $\emptyset \subset S$.
   (c) Since $\mathbb{Z} \subsetneq \mathbb{Q}$, there are a greater number of rationals than integers.
   (d) For $i := \sqrt{-1} \in \mathbb{C}$, $i > 0$.
   (e) If f is a function such that $f : x \mapsto y$, then $f^{-1} : y \mapsto x$. In other words, $y = f(x)$ iff $x = f^{-1}(y)$.

2. (20 points) Please prove that the function $f : \mathbb{R} \to \mathbb{R}$ defined as

$$f(x) = \begin{cases} x\sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

   is continuous at $x = 0$. **Hint:** This the same function considered in # 1 (b), p. 90, but here you are asked for a proof. Consider the definition of continuity. You may assume the standard properties of the sine function.

3. (20 points) Assume that $\{s_n\}$ and $\{t_n\}$ are sequences of reals, that $\{s_n\}$ is bounded, and that $t_n \to 0$ as n increases. Please prove that $s_n t_n \to 0$ as n increases. Note that the definition of convergence is at the top of p. 45 in our text.

4. (20 points) Recall that the definition of the *boundary* (p. 62) is $\partial S := \overline{S} \cap \overline{\mathcal{C}S}$. Please show that if p is a point on the boundary, than $\forall\ \epsilon > 0$, $\mathrm{B}_\epsilon(p)$ has a nonempty intersection with both S and $\mathcal{C}S$.

5. (20 points) Let (E, d) be a metric space. Suppose that $f(x, y) := 2d(x, y)$ and $g(x, y) := (d(x, y))^2$. Please prove or disprove each of the following statements:

   (a) (E, f) is a metric space.

   (b) (E, g) is a metric space.