MA3832 Name: Solutions

Final B Term, 2011

Show all work needed to reach your answers. You may use any theorem proven in our text, but cite any theorem that you use by name or description.

1. (15 points) Suppose that $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b). If $f' \equiv 0$, please explain why f is constant.

By the mean value theorem, $\forall x \in (a, b]$, $\exists c \in (a, x)$ s.t. $f(x) - f(a) = f'(c)(x-a)$. Since $f' \equiv 0$, $f(x) - f(a) = 0 \Rightarrow f(x) = f(a)$ $\forall x \in [a, b]$. $\blacksquare$

(-1): mishandle endpoints.

(-3): Show only that $f(b) = f(a)$.

(+5): Use definition rather than MVThm.

2. (15 points) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous. Please compute $F'(x)$ for

$$F(x) := \int_{x^2}^{x} f(t)\,dt$$

and please cite which theorem(s) allows you to compute this derivative.

Using the basic properties of integrals

$$F(x) = \int_0^x f(t)\,dt - \int_0^{x^2} f(t)\,dt$$

So by the fundamental theorem (+1) of Calculus and the chain (+1) rule,

$$F'(x) = \frac{d}{dx}\left(\int_0^x f(t)\,dt\right) - \frac{d}{dx}\left(\int_0^{x^2} f(t)\,dt\right)$$

$$= f(x) - f(x^2) \cdot \frac{d}{dx}(x^2)$$

$$= f(x) - 2x\, f(x^2)$$

(+4) (+2) (+3) (+4)

It is possible to solve this using the definition of the derivative.

$\blacksquare$

3. (20 points) Suppose that $\forall\ i \in \mathbb{Z}^+$, $a_i \in \mathbb{R}$. Please give ϵ–N definition for *convergence* for the series $\sum_{i=1}^{\infty} a_i$. **Hint**: What sequence should one consider?

(+4) Let $A_n = \sum_{i=1}^{n} a_i$ be the n^{th} partial sum. Then $\sum_{i=1}^{\infty} a_i = L$ iff the sequence of partial sums $\{A_n\}$ converges (+1) to L, i.e., given any (+4) $\epsilon > 0$, $\exists\ N \in \mathbb{Z}^+$ (+4) s.t. $|A_n - L| < \epsilon$ (+3) whenever $n > N$. (+4)

4. (20 points) Please explain why the function $f : \mathbb{R} \to \mathbb{R}$ (defined below) is integrable on the interval $[0, 1]$. You may assume that the sine function is continuous, and you may cite the appropriate homework problem.

$$f(x) := \begin{cases} \sin \dfrac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Since f is continuous (+8) on $(0,1]$ and bounded on (+8) $[0,1]$, we showed in the homework (+4) that f is Riemann integrable on $[0,1]$. Specifically, given any $\epsilon > 0$, $\exists\ \delta > 0$ s.t. (given a partition P) with $|P| < \delta < \frac{\epsilon}{4}$, then $|f(x)| < 1$ for $0 \le x \le |P|$ (the first subinterval) and $|f_u(x) - f_\ell(x)| < \epsilon/2$ $\forall\, x \in [|P|, 1]$ (uniform continuity on $[|P|, 1]$). So

$$\begin{aligned}\int_0^1 |f_u(x) - f_\ell(x)|\,dx &= \int_0^{|P|} |f_u(x) - f_\ell(x)|\,dx + \int_{|P|}^1 |f_u(x) - f_\ell(x)|\,dx \\ &\le 2|f(x)| \cdot |P| + \frac{\epsilon}{2}(1-0) \\ &< \epsilon/2 + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

$\blacksquare$

+8: mention continuity on $[0,1]$

+10: mention continuity on $(0,1]$, problem at $x = 0$.

+15: mention continuity on $(0,1]$, boundedness on $[0,1]$.

5. (20 points) Suppose that $\forall\, n \in \mathbb{Z}^+ \cup \{0\}$, $c_n \in \mathbb{R}$. For the power series $\sum_{n=0}^{\infty} c_n x^n$, please show that if r is the radius of convergence, then

$$r = \lim_{n\to\infty} \left| \frac{c_n}{c_{n+1}} \right|$$

provided this limit exists.

+15: convergence by the ratio test only:

Because the limit exists, given $\epsilon > 0$ (for convenience, $\epsilon < r$), $\exists\, N \in \mathbb{Z}^+$ s.t. $\forall\, n > N$,

$$r - \epsilon < \frac{|c_n|}{|c_{n+1}|} < r + \epsilon$$

or

$$\frac{|x|}{r+\epsilon} < \frac{|c_{n+1}||x|}{|c_n|} < \frac{|x|}{r-\epsilon} \quad \forall\, n > N.$$

+4 +8

So by the ratio test, if $|x| < r - \epsilon$, the power series converges, while if $r + \epsilon < |x|$, +8 it diverges. But since ϵ is arbitrary, r must be exactly the radius of convergence. ∎

6. (10 points) Please prove or give a counterexample: If a function $f : [a,b] \to \mathbb{R}$ is differentiable on (a,b) and strictly increasing on $[a,b]$, then $f'(x) > 0\ \forall\, x \in (a,b)$.

The problem is that while the difference quotient $\frac{f(x_1) - f(x_2)}{x_1 - x_2} > 0 \quad \forall\, x_1 \neq x_2$, the limit can converge to zero.

Counterexample: $f(x) = x^3$ on $[-1,1]$

+10

Then $f'(0) = 0$. ∎