MA3832 Name: *Solutions*

Final **D Term, 2002**

Show all work needed to reach your answers. You may use any theorem proven in our text, but cite any theorem by page number that you use.

1. (30 points) Recall the definition of $\ln(x)$:

$$\ln(x) := \int_1^x \frac{dt}{t}$$

(a) Please explain why $\ln(x)$ is positive for $x > 1$.

+10

$\ln(x)$ is positive for $x > 1$ because the integrand $\left(\frac{1}{t}\right)$ is positive. ($\ln(x)$ is the [positive] area under the curve $\frac{1}{t}$ from 1 to $x > 1$.)

(b) Please explain why $\ln(x)$ is increasing for $x > 0$.

+10

$\ln(x)$ is increasing because $\frac{d}{dx}(\ln(x)) = \frac{1}{x} > 0 \quad \forall x > 0.$

+10

$\ln(y) - \ln(x) = \ln\left(\frac{y}{x}\right)$ and $\frac{y}{x} > 1$ when $0 < x < y$.

+5 the derivative is positive

(c) Consider the function $f_n(x) := x^{1/n}$ for $n \in \mathbb{Z}^+$. Which grows faster, $\ln(x)$ or $f_n(x)$? Please justify your answer.

Working with the reciprocal is fine.

Consider $\lim\limits_{x \to \infty} \frac{\ln(x)}{x^{1/n}} = \lim\limits_{x \to \infty} \frac{\frac{1}{x}}{\frac{1}{n} x^{\frac{1}{n}-1}} = \lim\limits_{x \to \infty} \frac{n x^{1-\frac{1}{n}}}{x} = \lim\limits_{x \to \infty} n x^{-1/n} = 0$ (+1) (+1) (+1) (+1)

+4 +1 L'Hopital's Rule

So $f_n(x) = x^{1/n}$ grows faster $\forall n \in \mathbb{Z}^+$. +1

2. (25 points) Please use the definition of the derivative (for $x = 0$) and the rules for differentiation (for $x \neq 0$) to discuss the derivative of $f : \mathbb{R} \to \mathbb{R}$ defined as

$$f(x) = \begin{cases} x^2 \sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Is this function continuous? Does its derivative exist and is it continuous? Please explain your answer.

(A) +5 For $x \neq 0$, $f'(x) = 2x \sin\left(\frac{1}{x}\right) + x^2 \cos\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\right) = 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$

(B) +10 For $x = 0$, $f'(0) = \lim\limits_{h \to 0} \frac{h^2 \sin\left(\frac{1}{h}\right) - 0}{h} = \lim\limits_{h \to 0} h \sin(1/h) = 0$ since $|h \sin(\frac{1}{h})| \leq |h|$

So the derivative exists for all $x \in \mathbb{R}$. But f' is not continuous; +5 (C)

$f'(0) = 0$ but $\lim\limits_{x \to 0} f'(x)$ DNE.

Still, since f is differentiable on $\mathbb{R}$, it must also be continuous on $\mathbb{R}$.

+5 (D)

3. (15 points) Suppose that g is continuous on $[a,b]$ and differentiable on (a,b). If $g'(x) \neq 0 \ \forall\, x \in (a,b)$, please prove that f g is one-to-one on (a,b). **Hint**: The definition of one-to-one is on p. 10 of our text.

Suppose g is not 1-to-1. Then $\exists\, x_1, x_2 \in (a,b)$ $(x_1 \neq x_2)$ s.t. $g(x_1) = g(x_2)$.
But by the MV Thm, (A) $\exists\, \xi \in (x_1, x_2)$ s.t. $g(x_1) - g(x_2) = g'(\xi)(x_1 - x_2)$.
Since $x_1 \neq x_2$ but $g(x_1) = g(x_2)$, this implies $g'(\xi) = 0$, which contradicts that $g'(x) \neq 0 \ \forall x \in (a,b)$. So g is 1-to-1.

4. (15 points) For $a_n \in \mathbb{R}^+$, suppose that $\sum_{n=0}^{\infty} a_n$ converges absolutely. Please show that $\sum_{n=0}^{\infty} a_n^2$ also converges.

A (+4) B (+3)

Since $\sum_{n=0}^{\infty} a_n$ converges, $a_n \to 0$. So $\exists\, N > 0$ s.t. $\forall\, n > N$, $a_n < 1$.
C (+4) Thus $\forall\, n > N$, $0 < a_n^2 < a_n$, and hence $\sum_{n=0}^{\infty} a_n^2$ converges by the Comparison Test.

D (+4)

5. (15 points) Suppose that $T : \mathbb{R} \to \mathbb{R}$ is continuous. Please explain why the following equality holds $\forall\, x \in \mathbb{R}$:

$$\int_{-h}^{x} \int_{-h}^{t} T(s)\, ds\, dt \;=\; x \int_{-h}^{x} T(s)\, ds \;-\; \int_{-h}^{x} sT(s)\, ds$$

(+3) (A)

Notice that by the F Thm C and the product rule,

$$\frac{d}{dx}\left(\int_{-h}^{x}\int_{-h}^{t} T(s)\, ds\, dt\right) = \int_{-h}^{x} T(s)\, ds$$ (+3) (B)

while (+3) (C) (+2)

$$\frac{d}{dx}\left(x\int_{-h}^{x} T(s)\, ds - \int_{-h}^{x} sT(s)\, ds\right) = \int_{-h}^{x} T(s)\, ds + \left(xT(x) - xT(x)\right)$$

$$= \int_{-h}^{x} T(s)\, ds$$ (+1)

(+3) (D) So the two expressions (both sides) have the same derivative, and thus they differ by a constant. Since each term is zero at $x=-h$, that constant is zero and the two expressions are equal.