MA3832 Name: *Solutions.*

**Final** **D Term, 2003**

Show all work needed to reach your answers. You may use any theorem proven in our text, but cite any theorem by page number that you use.

1. (20 points) Suppose that $f, g : [a, b] \to \mathbb{R}$ are both Riemann integrable functions, and that $\exists\ m, M \in \mathbb{R}$ s.t. $m \leq f(x) \leq g(x) \leq M\ \forall\ x \in [a, b]$. What can be said about the integrals of $f$ and $g$?

+10

$$m(b-a) \leq \int_a^b f \leq \int_a^b g \leq M(b-a)$$

2. (30 points) Please discuss the continuity and differentiability at $x = 0$ of the functions $f, g : \mathbb{R} \to \mathbb{R}$ (defined below). Is each function itself continuous at $x = 0$? Does its derivative exist and is it continuous? Please explain your answer.

(a)

$$g(x) := \begin{cases} x\exp(\frac{1}{x}) & x > 0 \\ 0 & x \leq 0 \end{cases}$$

Notice that $g$ is not even (+5) continuous at $x=0$:

$$\lim_{x\to 0^+} x\exp(\tfrac{1}{x}) \overset{+5}{=} \lim_{x\to 0^+} \frac{\exp(1/x)}{1/x} = \lim_{x\to 0^+} \exp(\tfrac{1}{x}) = +\infty \neq 0.$$

$g$ not continuous at $x=0$ (+5) $\Rightarrow$ $g$ not differentiable at $x=0$.

(b)

$$f(x) := \begin{cases} x^2\sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Since $|\sin(\frac{1}{x})| \overset{+3}{\leq} 1\ \forall\ x \neq 0$, $\lim_{x\to 0} f(x) \overset{+3}{=} 0 = f(0)$, so $f$ is continuous at $x=0$. By definition

$f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} x\sin(\frac{1}{x}) = 0$. (+3) Using the product and chain rules,

$$f'(x) = 2x\sin(\tfrac{1}{x}) + x^2\cos(\tfrac{1}{x})(-\tfrac{1}{x^2})$$
$$= 2x\sin(\tfrac{1}{x}) - \cos(\tfrac{1}{x}) \quad \text{(+3)} \quad \text{for } x > 0.$$

Since $\lim_{x\to 0^+} f'(x)$ DNE, $f'$ is not continuous at $x=0$. (+3)

3. (40 points) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable, $f < 0$, $f$ is strictly increasing, and $\lim_{x\to\infty} f(x) = 0$.

(a) Suppose that $\lim_{x\to\infty} f'(x)$ exist and is finite. Please show that $\lim_{x\to\infty} f'(x) = 0$ (i.e., give an $\epsilon$–$M$ proof).

Suppose $\lim_{x\to\infty} f'(x) = L > 0$. Then given $\epsilon > 0$, $\exists\, M \in \mathbb{R}$ s.t. $|f'(x) - L| < \epsilon \quad \forall x > M$. So $L - \epsilon < f'(x) \quad \forall x > M$.

Pick any $x_1 > M$, and consider a second point $x_2 := x_1 - \frac{f(0)}{L-\epsilon} > x_1$.

By

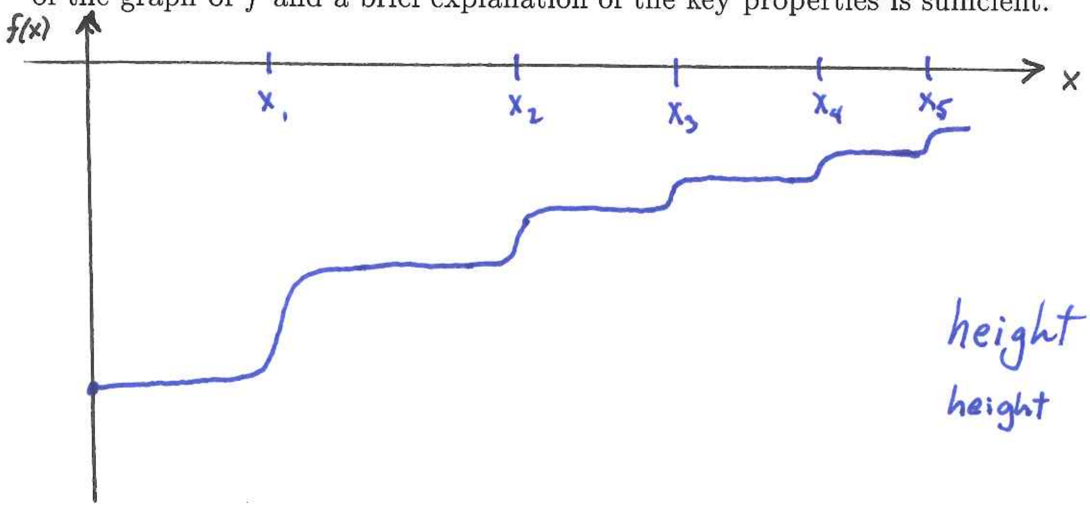
M the MVThm, $\exists\, \xi \in (x_1, x_2)$ s.t.

$$f(x_2) - f(x_1) = f'(\xi)(x_2 - x_1) > (L-\epsilon)\left(\frac{-f(0)}{L-\epsilon}\right) = -f(0)$$

So $f(x_2) > f(x_1) - f(0) > 0$, contradicting that $f < 0$. $\rightarrow\!\leftarrow$

Hence if $\lim_{x\to\infty} f'(x)$ exists, $\lim_{x\to\infty} f'(x) = 0$.

(b) Please give an example to show that the limit may not exist. **Hint**: A sketch of the graph of $f$ and a brief explanation of the key properties is sufficient.

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Let $f$ be an increasing function, close to a step function, with $f'(x_i) \geq 1$ at each step location $x_i$, but also let each step ~~size~~ height be less than half of the distance up to zero.

4. (10 points) Suppose that $a_1,\ a_2,\ a_3,\ \ldots\ a_n,\ \ldots$ is a *decreasing* sequence of positive reals. If

$$\sum_{n=1}^{\infty} a_n$$

converges, please show that $\lim_{n\to\infty} n\,a_n = 0$.

Since $\Sigma a_n$ converges, the Cauchy criterion applies (p. 142): Given $\epsilon>0$, $\exists N$ s.t. for $n/2 \geq N$ (here $n=2k$ is even),

$$a_{n/2} + a_{\frac{n}{2}+1} + a_{\frac{n}{2}+2} + \ldots + a_n < \epsilon/2$$

Since $\{a_n\}$ is decreasing, $a_k \geq a_n$ for $\frac{n}{2} \leq k \leq n$.

So

$$\left(\frac{n}{2}\right) a_n = \underbrace{a_n + a_n + a_n + \ldots + a_n}_{n/2} \leq a_{n/2} + a_{n/2+1} + \ldots + a_n < \epsilon/2$$

$\Rightarrow \quad n a_n < \epsilon$. So by definition $\lim_{n\to\infty} n a_n = 0$.