MA3832 Name: Solutions

Final **D Term, 2004**

Show all work needed to reach your answers. You may use any theorem proven in our text, but cite any theorem by name and/or page number that you use.

1. (20 points) Suppose that $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b). If $f' \equiv 0$, please explain why f is constant.

By the MVThm, (+4) for each $x \in (a,b]$, (+3) $\exists\, \xi_x \in (a,b)$ s.t. (+4)

$$\frac{f(x) - f(a)}{x - a} = f'(\xi_x) \quad (+5)$$

But $f'(\xi_x) = 0 \Rightarrow f(x) = f(a)$ (+4) $\forall x \in [a,b]$. Thus f is constant.

2. (25 points) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is uniformly continous, and then define $f_n(x) := f(x + \frac{1}{n})$. Please show that f_n is uniformly convergent (i.e., give an ϵ–δ proof).

Proof: Since f is uniformly continous, given $\epsilon > 0$, $\exists\, \delta > 0$ (+2) such that $\forall\, x, y \in \mathbb{R}$, if $|x - y| < \delta$ (+2), then $|f(x) - f(y)| < \epsilon$ (+4). (+1)

By LUB 2. (p. 26), $\exists\, N \in \mathbb{Z}^+$ (+3) such that $\dfrac{1}{N} < \delta$ (+1). So if $n > N$, (+1)

then $\frac{1}{n} < \delta$ (+2). Thus $\forall\, x \in \mathbb{R}$ and $\forall\, n > N$, $|x - (x + \frac{1}{n})| < \delta$ (+2) (+1)

implies $|f(x) - f(x + \frac{1}{n})| < \epsilon$ (+4). So $\forall x \in \mathbb{R}$ and $\forall n > N$, $|f(x) - f_n(x)| < \epsilon$ (+4)

which implies $f_n \to f$ uniformly.

3. (25 points) Please discuss the continuity and differentiability at $x = 0$ of the function $f : \mathbb{R} \to \mathbb{R}$ (defined below). Is each function itself continuous at $x = 0$? Does its derivative exist and is it continuous? Please explain your answer.

$$f(x) := \begin{cases} \dfrac{\sin x}{\sqrt[3]{x}} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Notice that $\lim_{x\to 0} \frac{\sin x}{\sqrt[3]{x}} = \lim_{x\to 0}\left[\frac{\sin x}{x}\cdot\frac{x}{\sqrt[3]{x}}\right]$

+9

$$= \lim_{x\to 0}\left[\frac{\sin x}{x}\, x^{2/3}\right]$$

$$= \left(\lim_{x\to 0}\frac{\sin x}{x}\right)\left(\lim_{x\to 0} x^{2/3}\right) \quad \left(\text{since both limits exist}\right)$$

$$= 1 \cdot 0 = 0$$

+1 +2

Thus $\lim_{x\to 0} f(x) = f(0) \Rightarrow f$ is continuous at $x=0$.

On the other hand, if f is differentiable at $x=0$,

+9

$$f'(0) = \lim_{x\to 0}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0}\frac{\sin x}{x^{4/3}} = \lim_{x\to 0}\left[\frac{\sin x}{x}\cdot\frac{1}{\sqrt[3]{x}}\right]$$

Since the first factor approaches 1, while the second is unbounded, this limit DNE. Thus $f'(0)$ DNE.

+1 +2

Of course, if $f'(0)$ DNE, f' can not be differentiable at $x=0$. +1

4. (20 points) Consider the two series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} (a_n)^n$. What can be said about the convergence/divergence relationship between these two series? If either one converges or diverges, must the other also converge or diverge? You may give a counterexample to show that one series can diverge while the other converges.

Counterexample (10)

Notice that if $a_n \equiv 1/2 \ \forall n$, then $\sum_{n=0}^{\infty} a_n$ diverges, but (+5)

$\sum_{n=0}^{\infty} (a_n)^n = \sum_{n=0}^{\infty} (\frac{1}{2})^n = \frac{1}{1-1/2} = 2$ (it's a geometric series). So

(+2) the divergence of $\sum_{n=0}^{\infty} a_n$ does not imply the divergence of $\sum_{n=0}^{\infty} a_n^n$

and automatically, the contrapositive (+1) also follows: $\sum_{n=0}^{\infty} a_n^n$ converges does not imply that $\sum_{n=0}^{\infty} a_n$ converges. (+2)

(+5)

On the other hand, suppose that $\sum_{n=0}^{\infty} a_n$ converges. Then $\lim_{n\to\infty} a_n = 0$, and hence $\exists N \in \mathbb{Z}^+$ s.t. $\forall n > N$, $|a_n| < 1/2$. So $\sum_{n=0}^{\infty} |a_n|^n$ converges by comparison to the geometric series $\sum_{n=0}^{\infty} (\frac{1}{2})^n$, and this implies that $\sum_{n=0}^{\infty} a_n^n$ is absolutely convergent, (+2) thus convergent. The (+1) contrapositive again holds: If $\sum_{n=0}^{\infty} a_n^n$ diverges, then $\sum_{n=0}^{\infty} a_n$ must also diverge. (+2)

5. (10 points) Consider the following indefinite integral (antiderivative):

$$2\int \cos x \sin x \, dx$$

On seeing this integral, one student makes the substitution $u = \cos x \Rightarrow du = -\sin x \, dx$, and, using the fundamental theorem of calculus, finds that the value of this integral is $-\cos^2 x + C$. But another student makes the substitution $u = \sin x \Rightarrow du = \cos x \, dx$, and finds that the value of this integral is $\sin^2 x + C$. Has either student made a mistake? Please explain why or why not.

+1 +3

Both students are correct. Antiderivatives are only defined up to additive constants, so if one defines $f(x) = -\cos^2 x + C_1$ and $g(x) = \sin^2 x + C_2$, one sees that

$$f(x) - g(x) = (-\cos^2 x + C_1) - (\sin^2 x + C_2) = -\cos^2 x - \sin^2 x + C_1 - C_2$$

$$= -1 + C_1 - C_2$$

$\Rightarrow$ f and g differ by a constant.

+6