MA3832 Name: *Solutions*

Quiz 3 B Term, 2011

Show all work needed to reach your answers.

1. (10 points) Suppose that $F : (\alpha, \beta) \to \mathbb{R}$, that $F(1) = -1$, and that $F'(x) = f(x)$ where $f : (\alpha, \beta) \to \mathbb{R}$ is continuous.
   (a) According to the Fundamental Theorem of Calculus, what definite integral has f as its derivative and is zero at $x = 1$?

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$$G(x) = \int_1^x f(t)\,dt$$

   (b) Call the integral from the previous question $G(x)$. What is the value of $F(2) - G(2)$? Please explain your answer.

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By the Mean Value Theorem, since $F'(x) \overset{+1}{=} G'(x) = f(x)$ $\forall x \in (\alpha, \beta)$, $F(x) \overset{+1}{-} G(x)$ is constant. So $F(2) - G(2) = F(1) \overset{+3}{-} G(1) = -1 - 0 = -1$

2. (15 points) Suppose that $f : [0,1] \to \mathbb{R}$ is an increasing function. Please show that f is integrable. **Hint**: Consider defining upper and lower step functions whose integrals are arbitrarily close to each other. For all $x \in [0,1]$, start with $f_0^u(x) := f(1)$ and $f_0^\ell(x) := f(0)$.

Given any $\epsilon > 0$, let $P = \{0, \frac{1}{N}, \frac{2}{N}, \cdots, \frac{N-1}{N}, 1\}$ (+2) where $|P| = \frac{1}{N} \overset{+2}{<} \delta := \frac{\epsilon}{f(1) - f(0)}$.
So this is a uniform (+1) partition, and for this partition,

$$f_u(x) := \begin{cases} f(\frac{i}{N}) & x \in (\frac{i-1}{N}, \frac{i}{N}),\ 1 \le i \le N \\ f(\frac{i}{N}) \ (+1) & x = i/N,\ 0 \le i \le N \end{cases} \quad \text{and} \quad f_\ell(x) := \begin{cases} f(\frac{i-1}{N}) & x \in (\frac{i-1}{N}, i/N),\ 1 \le i \le N \\ f(\frac{i-1}{N}) \ (+1) & x = i/N,\ 0 \le i \le N \end{cases}$$

Then $f_\ell(x) \le f(x) \overset{+1}{\le} f_u(x)$ $\forall x \in [0,1]$ and

$$\int_0^1 (f_u(x) \overset{+1}{-} f_\ell(x))\,dx \overset{+1}{=} \int_0^1 \sum_{k=1}^{N} \left(f(\tfrac{i}{N}) - f(\tfrac{i-1}{N})\right) \Delta x\,dx \overset{\text{telescopes}\ (+2)}{=} \int_0^1 (f(1) - f(0))\,\Delta x\,dx$$
$$= (f(1) - f(0))\frac{1}{N} \overset{+1}{<} \epsilon.$$

The existence of these upper and lower step functions that are arbitrarily close together (+2) $\Rightarrow$ f is integrable.