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Approximately Periodic Solutions of the Elastic String Equations

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Abstract Glimm's method is used to compute solutions of initial-boundary value problems for the nonlinear system of equations describing the motion of an elastic string. For certain initial data, numerical results suggest the existence of a stable, approximately periodic solution containing no shocks. This solution consists of two new exact solutions which are patched together by the numerical algorithm.

KEY WORDS: Elastic string, periodic solutions, numerical solutions.
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INTRODUCTION

Consider an elastic string in the plane whose configuration in $\mathbf{R}^2$ at time t is specified by a function $\mathbf{r}(\cdot,t) : [-1,1] \to \mathbf{R}^2$. The interval $[-1,1]$ is the *reference configuration*, each point of which identifies a material point in the string. Let $\xi = |\mathbf{r}_x(x,t)|$ be the local elongation, and let $T(\xi)$ be the tension, where $T : (0,\infty) \to \mathbf{R}$ is a smooth function. In the absence of external forces, assuming constant density and fixed endpoints a distance L apart, the initial boundary value problem for the string is formulated as follows[1].

$$
(P)\begin{cases}
\left[\frac{T(\xi)}{\xi}\mathbf{r}_x(x,t)\right]_x = \mathbf{r}_{tt}, \quad -1 < x < 1,\ t > 0, & (1.1)\\
\mathbf{r}(-1,t) = (0,0), \qquad \mathbf{r}(1,t) = (L,0),\ t > 0, & (1.2)\\
\mathbf{r}(x,0) = \mathbf{f}(x), \qquad \mathbf{r}_t(x,0) = \mathbf{g}(x),\ -1 < x < 1. & (1.3)
\end{cases}
$$

In this paper, the tension is taken to satisfy

$$
\begin{aligned}
&\text{(a)}\quad T(1) = 0,\\
&\text{(b)}\quad T'(\xi) > T(\xi)/\xi > 0, \qquad 1 \le \xi \le \xi_{\max}, \qquad (1.4)\\
&\text{(c)}\quad T''(\xi) < 0, \qquad 1 < \xi < \xi_{\max}.
\end{aligned}
$$

Conditions (1.4) guarantee that (1.1) is a strictly hyperbolic system for $1 < \xi < \xi_{\max}$, with two genuinely nonlinear characteristic families and two linearly degenerate characteristic families. For numerical calculations, the tension function $T(\xi) = 4\ell n\xi$, with $\xi_{\max} = e$, was used.

In §2, we present two families of new exact solutions of (1.1). (Other explicit solutions have been given by Rosenau and Rubin[9,10]. Each of our solutions corresponds to a vertical motion of the string. In the first solution, the string is straight and accelerates around a single fixed point (cf. Fig.1a). In the second, the string is curved and its shape does not change in time; the string accelerates due to its curvature and nonzero tension (cf. Fig.1b). The significance of these solutions is that when they are combined to form a periodic function by matching, the periodic function is an approximate solution of (1.1), (1.2) that resembles numerical solutions that are approximately periodic.

Numerical solutions are obtained from a deterministic version of Glimm's numerical method[4]. In Glimm's method, the Riemann problem plays a central role. (The Riemann problem for (P) consists of (1.1), (1.3), with $\mathbf{f}'(x)$ and $\mathbf{g}(x)$ constant on each side of a single jump discontinuity.) For large systems, it is generally inefficient to compute solutions of Riemann problems, even when the Riemann problem has been solved analytically. For the elastic

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2. Exact Solutions

In this section, we describe in detail

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string, however, the solution is computed by a simple algorithm outlined in Shearer[11]*. Glimm's method displays shocks sharply, and can therefore be used to detect the formation of shock waves.

In the numerical experiments described in §3, we first choose initial data which would yield a mode of vibration for the linear wave equation, and observe the formation of shock waves in finite time, as expected for the nonlinear problem. However, with somewhat different initial data, persistent oscillations of the string are observed, which are interpreted as numerical evidence of a time periodic solution of (P). The structure of these approximately periodic numerical solutions led to the discovery of the exact solutions of system (1.1) described in §2.

The approximately periodic numerical solution combines the explicit solutions as follows. Near its fixed endpoints, the string is straight and rotates; the central portion of the string is curved and accelerating. When the string is at its maximum amplitude of vibration, the boundaries between the two types of solutions are at the endpoints, and the curved central solution extends along the entire string. As the string moves towards a straight horizontal configuration, the boundaries move inward towards the center of the string. The entire solution depends upon a single parameter, which is roughly the amplitude of the vibration. Analytically, the two explicit solutions match only to first order in the amplitude, but the numerical solution seems to adjust for the higher order discrepancies by allowing small horizontal velocities, without the adjustments growing in time or leading to shock formation.

2. Exact Solutions

In this section, we describe in detail two families of exact solutions involving

*Earlier solutions of Riemann problems were given by Mihailescu and Suliciu [7,8]. These papers have been overlooked in the recent literature.

only vertical motion of the string. We then combine the exact solutions to construct a time periodic approximate solution of (1.1), (1.2). It is convenient to write equation (1.1) as a 4×4 first order system. Let $\mathbf{U} = (p, q, u, v) = (\mathbf{r}_x, \mathbf{r}_t)$, and let $\xi = (p^2 + q^2)^{\frac{1}{2}}$. Then (1.1) becomes

$$\mathbf{U}_t = \mathbf{F}(\mathbf{U})_x, \tag{2.1}$$

where $\mathbf{F}(p, q, u, v) = (u, v, pT(\xi)/\xi,\ qT(\xi)/\xi)$.

Theorem 1: Equation (2.1) admits two families of solutions with zero horizontal velocity:

(i) If $\mathbf{U} = \mathbf{U}_b(x,t) = (p(x,t), q(t), 0, v(x))$ is a C^1 solution of (2.1), then there are constants p_0, q_0, v_0, and c such that

$$\mathbf{U}_b(x,t) = (p_0, q_0 + ct, 0, v_0 + cx). \tag{2.2}$$

(ii) If $\mathbf{U} = \mathbf{U}_m(x,t) = (p(x,t), q(x), 0, v(t))$ is a C^1 solution of (2.1) with $q(0) = 0$, then there are constants α and v_1, and $\beta > 0$ such that

$$\mathbf{U}_m(x,t) = (Q(x; \alpha, \beta), \alpha x Q(x; \alpha, \beta), 0, v_1 + \alpha\beta t) \tag{2.3}$$

where

$$Q(x; \alpha, \beta) = \frac{T^{-1}\left(\beta\sqrt{1 + (\alpha x)^2}\right)}{\sqrt{1 + (\alpha x)^2}}. \tag{2.4}$$

Proof: (i) Substitute $\mathbf{U}(x,t) = (p(x,t), q(t), 0, v(x))$ into (2.1). The forms of q and v follow immediately since $q_t = v_x$. Also $p_t = u_x = 0$. Now the third component of (2.1) becomes

$$\left[\frac{T(\xi)}{\xi} p\right]_x = \left[\left(T'(\xi) - \frac{T(\xi)}{\xi}\right)\frac{p^2}{\xi^2} + \frac{T(\xi)}{\xi}\right] p_x = u_t = 0.$$

By assumption (1.4), the large bracketed expression is always positive; thus $p_x = 0$. Similarly, the fourth component of (2.1) is trivially satisfied since $q_x = v_t = 0$.

APPROXIMATELY

(ii) Substitute $\mathbf{U}(x,t) = (p(x$
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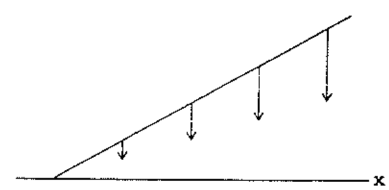

FIGURE 1 Two exact sol
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Also $p_t = u_x = 0$. Now the third

$\frac{T(\xi)}{\xi}\Big] p_x = u_t = 0.$

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of (2.1) is trivially satisfied since

(ii) Substitute $U(x,t) = (p(x,t), q(x), 0, v(t))$ into (2.1). Again $p_t = u_x = 0$. The fourth component of (2.1) implies that $v(t) = v_1 + ht$ and, considering the normalization $q(0) = 0$, that $\frac{T(\xi)}{\xi} q = hx$, where v_1 and h are constant. Since $\frac{T(\xi)}{\xi} p = \beta$ is constant and $\xi^2 = p^2 + q^2$, the expressions for p and q follow with $h = \alpha\beta$. The proof is complete.

Remark: For (ii) above, the assumption that $q(0) = 0$ implies that p is even in x and that q is odd. Hence the string is symmetric about $x = 0$. More generally, one could drop this assumption and consider a nonsymmetric string; then $\frac{T(\xi)}{\xi} q = hx + \gamma$ where γ is an additional constant. Then U_b and U_m both constitute four-parameter families of solutions.

The solution $U = U_b(x,t)$ of (2.2) satisfies the left boundary condition of (1.2) when $v_0 = c$ and the right boundary condition when $v_0 = -c$. With v_0 fixed at either of these values, this solution represents a straight string swinging about the endpoint with each point on the string moving vertically (cf. Fig. 1a). On the other hand, $U = U_m(x,t)$ given by (2.3) represents a uniformly accelerating string having convex or concave shape, cf. Fig. (1b).

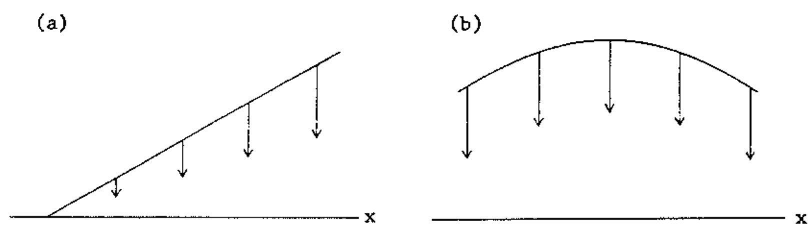

FIGURE 1 Two exact solutions. In (a) the string is contracting and swinging about a fixed point. The velocity is increasing in x and constant in time. In (b) the string is accelerating uniformly in time ($\alpha < 0$ for the configuration shown).

The acceleration results from the net force of tension (due to curvature) on the string segment.

A time periodic function $\mathbf{U}_p$ can now be created by matching $\mathbf{U}_b$ and $\mathbf{U}_m$. To do so, take $\mathbf{U}_p = \mathbf{U}_b$ near each endpoint, $\mathbf{U}_p = \mathbf{U}_m$ near the middle of the string, then match the three functions to lowest order in the parameters. One finds the following function $\mathbf{U}_p(x,t;a)$, defined over a quarter period $0 \le t \le \sqrt{\xi_0/T(\xi_0)}$ as

$$\mathbf{U}_p(x,t;a) = \begin{cases} \mathbf{U}_L(x,t;a), & -1 < x < t\sqrt{T(\xi_0)/\xi_0} - 1 \\ \mathbf{U}_M(x,t;a), & t\sqrt{T(\xi_0)/\xi_0} - 1 < x < 1 - t\sqrt{T(\xi_0)/\xi_0} \\ \mathbf{U}_R(x,t;a), & 1 - t\sqrt{T(\xi_0)/\xi_0} < x < 1 \end{cases}$$

where, in (p,q,u,v) coordinates,

$$\begin{aligned} \mathbf{U}_L(x,t;a) &= \xi_0\big(1, a(1 - t\sqrt{T(\xi_0)/\xi_0}), 0, -a\sqrt{T(\xi_0)/\xi_0}(1+x)\big), \\ \mathbf{U}_M(x,t;a) &= \big(Q(x;a,T(\xi_0)), -axQ(x;a,T(\xi_0)), 0, -aT(\xi_0)t\big), \\ \mathbf{U}_R(x,t;a) &= \xi_0\big(1, a(t\sqrt{T(\xi_0)/\xi_0} - 1), 0, a\sqrt{T(\xi_0)/\xi_0}(x-1)\big), \end{aligned}$$

and $Q(x;\alpha,\beta)$ is given by (2.4).

Remarks:

(i) The function $\mathbf{U}_p(x,t;a)$ satisfies (2.1) away from $x_c = \pm(1 - t\sqrt{T(\xi_0)/\xi_0})$. The matching at these points is exact for the velocities and for the slope of the string, q/p. However, the elongation $\xi = (p^2+q^2)^{1/2}$ (and hence the tension) experiences a jump of magnitude $O(a^2)$ at x_c. Thus $\mathbf{U}_p(x,t;a)$ is *not* a weak solution of (2.1), since there is no corresponding jump in longitudinal velocity. Note that $\mathbf{U}_p$ is set up to satisfy the fixed endpoint boundary conditions (1.2): $\mathbf{r}_t(\pm1,t) = 0$.

(ii) At $t = 0$, the string starts from rest with slope $q/p = -ax$. At $t = \sqrt{\xi_0/T(\xi_0)}$,

$$\mathbf{U}_p(x,t;a) = \begin{cases} (\xi_0, 0, 0, -a\sqrt{T(\xi_0)\xi_0}(1+x)), & -1 < x < 0 \\ (\xi_0, 0, 0, -a\sqrt{T(\xi_0)\xi_0}(1-x)), & 0 < x < 1 \end{cases},$$

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$$\mathbf{U}_p(x,t;a) = \begin{cases} \mathbf{U}_L(x,t;a), \\ \tilde{\mathbf{U}}_M(x,t;a), \\ \mathbf{U}_R(x,t;a), \end{cases}$$

where

$$\tilde{\mathbf{U}}_M(x,t;a) = \big(Q(x;a,T(\xi_0)), ax$$

The period of $\mathbf{U}_p$ is $4\sqrt{\xi_0/T(\xi_0)}$.
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3. Numerical Results:

In this section, we discuss numerica
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so that the string is horizontal, and the middle solution $\mathbf{U}_m$ has disappeared. The equilibrium length L of the string is related to ξ_0 by $L = 2\xi_0$.

(iii) While ξ_0 is determined by the equilibrium length of the string, the second parameter a is arbitrary and will be referred to as the *amplitude.* In particular, for $a = 0$, Eq. (2.3) reduces to the equilibrium solution

$$\mathbf{U}_p(x,t;0) = (\xi_0, 0, 0, 0). \tag{2.5}$$

(iv) The function $\mathbf{U}_p(x,t;a)$ may be extended for all time by shifting and reversing time in the obvious fashion. For example, for the next quarter period, $\sqrt{\xi_0/T(\xi_0)} \le t \le 2\sqrt{\xi_0/T(\xi_0)}$

$$\mathbf{U}_p(x,t;a) = \begin{cases} \mathbf{U}_L(x,t;a), & -1 < x < 1 - t\sqrt{T(\xi_0)/\xi_0} \\ \tilde{\mathbf{U}}_M(x,t;a), & 1 - t\sqrt{T(\xi_0)/\xi_0} < x < t\sqrt{T(\xi_0)/\xi_0} - 1 \\ \mathbf{U}_R(x,t;a), & t\sqrt{T(\xi_0)/\xi_0} - 1 < x < 1 \end{cases}$$

where

$$\tilde{\mathbf{U}}_M(x,t;a) = \left(Q(x;a,T(\xi_0)), axQ(x;a,T(\xi_0)), 0, aT(\xi_0)(t - 2\sqrt{\xi_0/T(\xi_0)})\right).$$

The period of $\mathbf{U}_p$ is $4\sqrt{\xi_0/T(\xi_0)}$. The graph of $\mathbf{U}_p$ is shown in Fig. 2 over a full period.

(v) By linearizing (2.1) about the equilibrium solution (2.5), one obtains the classical wave equation modeling small vertical vibrations of the string. If $\mathbf{U}_p$ is expanded in a, the lowest order terms are the solution of the classical wave equation. Also the period of $\mathbf{U}_p$ is the period of the principal mode of vibration for the linearized solution.

3. Numerical Results:

In this section, we discuss numerical experiments conducted on the initial boundary value problem (P) using the deterministic version of Glimm's

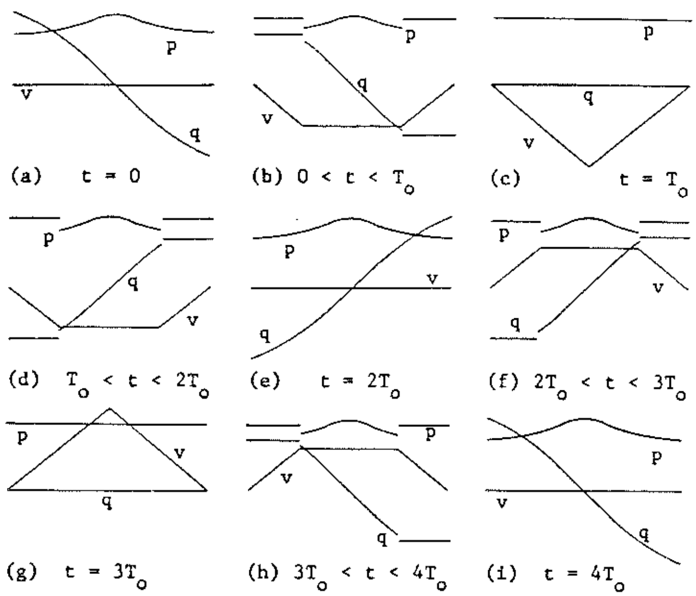

FIGURE 2 One full period for $\mathbf{U}_p$. In (a), (b) and (c), $\mathbf{U}_p$ is simply $\mathbf{U}_m$; in (c) and (g), $\mathbf{U}_p$ is only $\mathbf{U}_L$ and $\mathbf{U}_R$. The other parts of the figure show the transition from one type of solution to the other. Note that the matching for p and q is not exact. $T_0 = \sqrt{\xi_0/T(\xi_0)}$

method. A detailed discussion of the particular version of Glimm's method used is given in Fehribach[3]. It is similar to the deterministic version of Glimm's scheme proposed by Colella[2], in that it is based on a van der Corput sequence to define the sampling points. The interval $[-1,1]$ is divided into 100 subintervals, so that the spacing between Riemann problem grid points is $\Delta x = 1/100$; for the tension function $T(\xi) = 4\ell n\xi$, $1 \le \xi \le e$,

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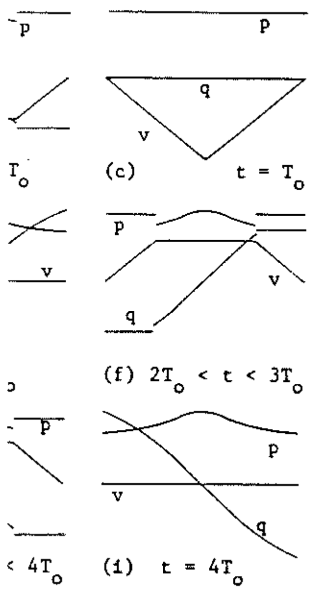

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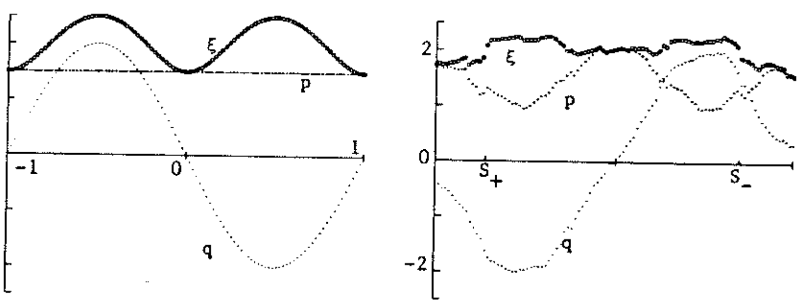
the choice $\Delta t = \Delta x/3$ satisfies the Courant Friedrichs Lewy condition. In Fehribach[3], we give numerical results for sample problems designed to test the code, and to explore simple properties of the equations, such as binary wave interactions, shock formation, and repeated reflections of waves at the endpoints. Here, we discuss numerical results for the initial boundary value problem, with various choices of initial data, to illustrate shock formation and to demonstrate numerical evidence of a stable time periodic solution.

To discuss shock formation, we first consider the families of characteristic fields. These are described in detail in[11,12]. There are two families of fields, each with two characteristic speeds that differ only in sign. One family consists of longitudinal waves; across these waves, the elongation (and tension) changes, but the slope remains constant. These waves satisfy the scalar nonlinear wave equation obtained from (1.1) by taking $\mathbf{r}_t$ and $\mathbf{r}_x$ to be proportional to a constant vector. Longitudinal waves are genuinely nonlinear under condition (1.4). The second family of characteristic fields corresponds to waves in which the slope of the string varies while the elongation remains constant. These are the transverse waves; they are linearly degenerate. In linearizing system (1.1) about an equilibrium solution, it is the transverse wave family that gives rise to the familiar linear wave equation used to approximate small amplitude transverse vibration.

Since shock waves in a solution imply the dissipation of energy, a periodic solution necessarily would have no shocks. Keller and Ting[5] have shown that there exists no purely longitudinal periodic solution of (1.1), since purely longitudinal motion leads to the formation of shocks. When transverse motion is included, one still might expect longitudinal waves to develop, grow, and lead to shocks. However, no analytic results cover this situation precisely. It is known that for small compactly supported initial data, provided the longitudinal component of the initial data is sufficiently large in comparison to the transverse component, then shocks form in finite time [6], but the analysis depends on wave interactions taking place over a limited time. For the initial boundary value problem, the wave interactions continue indefinitely, and the mechanism of shock formation is more subtle.

Computations shown in Fig. 3 illustrate shock formation. (Note that only p, q and ξ are shown.) The initial data $\mathrm{U}(x,0) = (1.5, -2\sin\pi x, 0, 0)$ is shown in Fig. 3a. Note that the horizontal component of the local elongation is initially constant. This initial data, if posed for the linear wave equation, would produce the first harmonic of the fundamental mode. In Fig. 3b ($t \simeq 1.707$), small discontinuities appear in ξ, indicating the formation of shocks in the longitudinal family.



(a) t = 0 (b) t ≈ 1.707 (512 iterations)

FIGURE 3 Shock formation. The initial data is shown in (a). In (b) the discontinuities in ξ imply the formation of shocks.

While the special functions $\mathrm{U}_p(x,t;a)$ of §2 can be matched only to lowest order in the amplitude a, numerical evidence indicates that a family of approximately periodic solutions, parameterized by amplitude and resembling $\mathrm{U}_p(x,t;a)$, exists for large amplitudes and persists for surprisingly long times. The numerical algorithm seems to patch smoothly together the

exact solutions given in Theorem
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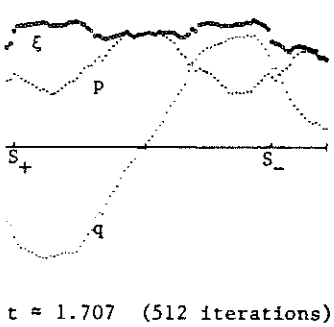

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exact solutions given in Theorem 1, and errors in the patching do not appear to grow with time. These approximately periodic solutions differ from $\mathbf{U}_p(x,t;a)$ in two ways: The horizontal velocity for these solutions, though small, is distinctly nonzero. Also the period of oscillation is somewhat less than the period for the corresponding $\mathbf{U}_p(x,t;a)$. Numerically both the horizontal velocity and the correction in the period of oscillation appear to be second order in the amplitude a. These results are consistent with the perturbation expansion given for periodic solutions of (1.1), (1.2) by Keller and Ting[5]. In this expansion both quantities are exactly second order in amplitude.

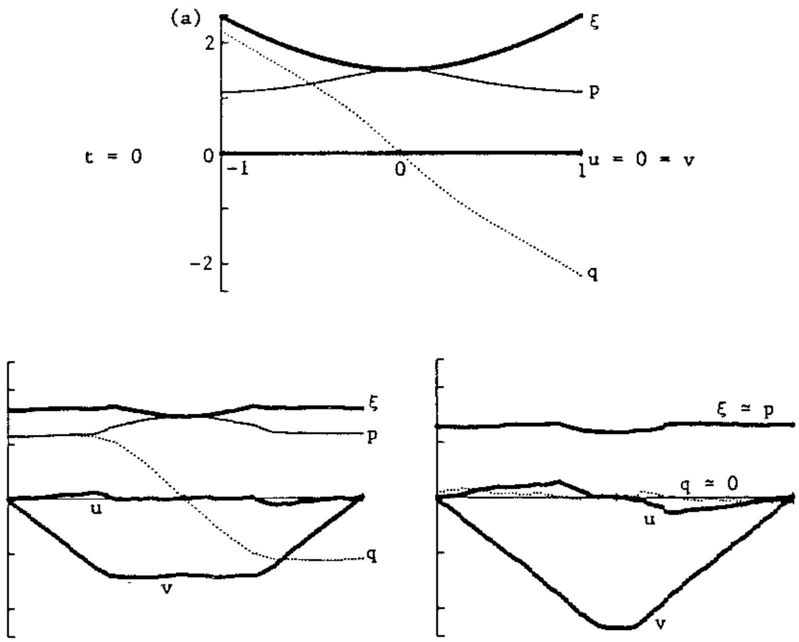
In Fig. 4, we display numerical results using 100 subintervals with $\Delta t = \Delta x/3$. The initial data (shown in Fig. 4a) is

$$\mathbf{U}(x,0) = (Q(x;2,T(1.5)),\ 2xQ(x;2,T(1.5)),0,0).$$

This, of course, is exactly $\mathbf{U}_M(x,0,-2)$ with $\xi_0 = 1.5$. Note that the relative sizes of these configurations are similar to those in Fig. 3. Figure 4a-c shows one quarter of a period. The period is approximately 3.5. In comparison, the period for $\mathbf{U}_p(x,t;2)$ with $\xi_0 = 1.5$ is 3.84679. Also note that the maximum value of u appears to occur near $t = 3$; this value is approximately 1/2. Calculations for this configuration have been carried through four oscillations, and although numerical noise sets in, the pattern and the period of oscillation remain unchanged. Recall that in Fig. 3, shocks had formed by $t \simeq 1.707$.

As a second example, consider the initial-boundary value problem shown in Fig. 5. These calculations were carried out on an IBM 3081 (all previous calculations were performed on PC's). Again 100 subintervals were used. The initial data is

$$\mathbf{U}(x,0) = (2,x,0,0)$$

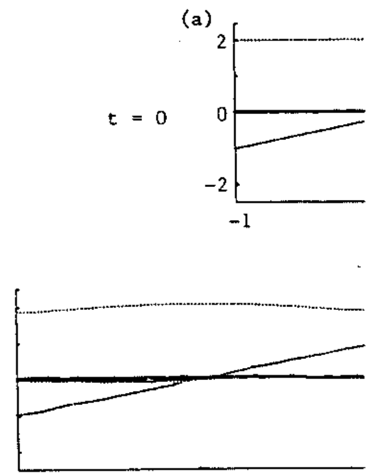


(b) t ≃ 0.4267 (128 iterations) (c) t ≃ 0.8533 (256)

FIGURE 4 The first quarter period of an approximately periodic solution. The initial conditions representing a convex string held above the horizontal axis are given in (a). In (b) the string is accelerating downward. In (c) it is nearly horizontal with the middle section moving downward with maximum speed.

which gives the string the initial parabolic profile $\mathbf{r}(x,0) = (2x+2,\ \frac{1}{2}(x^2-1))$. This data corresponds approximately to $\mathbf{U}_p(x,t;-1)$ with $\xi_0 = 2$; the period for this $\mathbf{U}_p$ is 3.39729. The maximum value of u observed in this run lies

between 0.1 and 0.15. The differ
and that of the previous example
amplitude. However, the correctio
to measure. By comparing Fig. 5a



(b) t ≃ 3.413 (1024 iterati

FIGURE 5 A second ap
conditions are given in (a).
below the horizontal axis. I
complete oscillations respec

the numerical solution is approxin
past the initial configuration.) No
complete oscillations respectively.
are obscured by numerical noise.
reduce this noise and suggest tha

between 0.1 and 0.15. The difference between this maximum value for u and that of the previous example is consistent with u being second order in amplitude. However, the correction to the period is so small as to be difficult to measure. By comparing Fig. 5a, 5b, and 5c, one sees that the period of

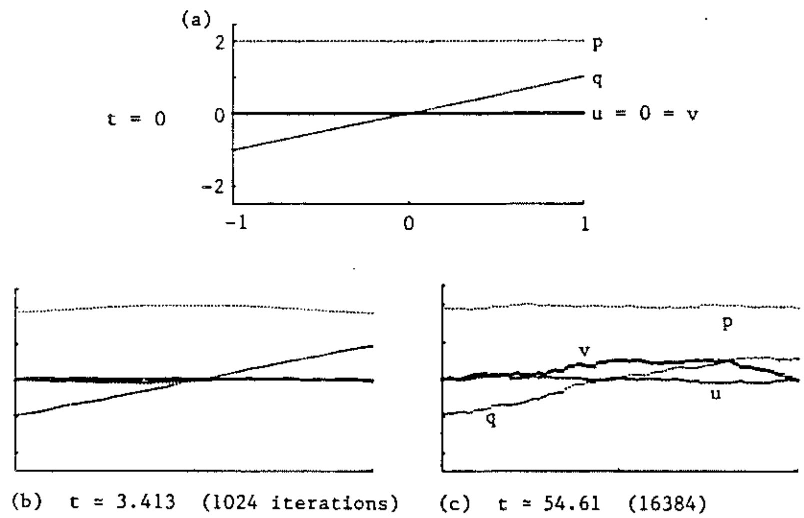


FIGURE 5 A second approximately period solution. The initial conditions are given in (a). Here the string is initially held concaved below the horizontal axis. Plots (b) and (c) represent one and sixteen complete oscillations respectively.

the numerical solution is approximately 3.4. (In Fig. 5c, the string is slightly past the initial configuration.) Note that Fig. 5b and 5c represent 1 and 16 complete oscillations respectively. The patterns continue to repeat until they are obscured by numerical noise. Calculations made using 300 subintervals reduce this noise and suggest that the oscillations persist indefinitely.

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REFERENCES

1. S.S. Antman, The equations for large vibrations of strings, Amer. Math. Monthly **87**, 359-370, (1980).
2. P. Colella, Glimm's method for gas dynamics, SIAM J. Sci. Stat. Comput. **3**, 76-110, (1982).
3. J. Fehribach and M. Shearer, The elastic string equations: numerical results using Glimm's method and two new exact solutions, CRSC Report, North Carolina State University, (1987).
4. J. Glimm, Solutions in the large for nonlinear hyperbolic systems of equations, Comm. Pure Appl. Math. **18**, 95-105, (1965).
5. J.B. Keller and L. Ting, Periodic vibrations of systems governed by nonlinear partial differential equations, Comm. Pure Appl. Math. **19**, 371-420, (1966).
6. T-P. Liu, Development of singularities in the nonlinear waves for quasi-linear hyperbolic partial differential equations, J. Differential Equations **33**, 92-111, (1979).
7. M. Mihailescu and I. Suliciu, Riemann and Goursat step data problems for extensible strings with nonconvex stress-strain relation, Rev. Roum. Math. Pures Appl. **20**, 551-559, (1975).
8. M. Mihailescu and I. Suliciu, Riemann and Goursat step data problems for extensible strings, J. Math. Anal. Appl. **52**, 20-24, (1975).
9. P. Rosenau and M.B. Rubin, Motion of a nonlinear string: Some exact solutions to an old problem, Phys. Review **31**, 3480-3482, (1985).
10. P. Rosenau and M.B. Rubin, Some nonlinear three-dimensional motions of an elastic string, Physica **17D** (1985).
11. M. Shearer, The Riemann problem for the planar motion of an elastic string, J. Differential Equations **61**, 149-163, (1986).
12. M. Shearer, The interaction of transverse waves for the vibrating string, Physical Mathematics and Nonlinear Partial Differential Equations, (ed. J.H. Lightbourne and S.M. Rankin), Marcel Dekker, (1985).

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Local Existence a Quasilinear Hy Problem

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Abstract An initial-bou
linear hyperbolic equatic
ary information, an unknc
tion is determined and an
small times is proved

KEY WORDS: Inverse problems, n
rations of an elastic

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1. INTRODUCTION

We consider the following quas

$$u_{tt}(x,t) - q(\int_0^L |u_x(y,t)|$$

with the initial and boundary

$$u(x,0) = u_0(x) ,$$

$$u_t(x,0) = u_1(x) ,$$

$$u(0,t) = u(L,t) = 0 ,$$

(*) The author is a member of