MA4473 Name: Solutions

Final, Part 1 (closed book, closed notes) **D Term, 2005**

1. (20 points) Please define/describe/state each of the following:

   (a) Neumann Problem $\Delta u = 0 \quad x \in \Omega$
   $\partial_n u = 0 \quad x \in \partial\Omega$

   (b) Poisson Equation
   $\Delta u = f \boxed{\neq 0}$ +1

   (c) Test Function $\varphi \in C_0^\infty(\Omega)$

   (d) Similarity Solution
   For a PDE with two independent varibles, a solution where these varibles appear only in a certain combination, e.g., $\xi = \frac{x}{\sqrt{t}}$

2. (10 points) Suppose a harmonic function u is defined on a bounded domain Ω. What does the maximum principle guarantee?
   That $\max_{x\in\Omega}(u) \subset \partial\Omega$ and $\boxed{\min_{x\in\Omega}(u) \subset \partial\Omega}$ +2 ($\max_{x\in\Omega}(u)$ and/or $\min_{x\in\Omega}(u)$ are in Ω° iff u is constant on $\overline{\Omega}$). +1

3. (20 points) Consider the PDE $a(x,y)u_x + b(x,y)u_y = 0$. According to the chain rule, what equations involving a variable s must be satisfied if u is to be constant with respect to s? Explain your answer.
   By the chain rule, we need
   $$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} = a(x,y)u_x + b(x,y)u_y = 0$$
   Thus u is constant w.r.t. s only if

   Equations: $\frac{\partial x}{\partial s} = a(x,y)$

   $\frac{\partial y}{\partial s} = b(x,y)$

MA4473 Name: *Solutions*

Final, Part 2 (open book, open notes) **D Term, 2005**

Show all work needed to reach your answers. You may use any theorem we discussed in the course, but please cite any result you use.

1. (25 points) Consider the following heat-flow system:

$$\begin{array}{rll} \text{PDE:} & u_t = \kappa \triangle u & \boldsymbol{x} \in \Omega, \quad t > 0 \\ \text{BC:} & u(\underline{x}, t) = 0 & \boldsymbol{x} \in \partial\Omega, \ t > 0 \\ \text{IC:} & u(\underline{x}, 0) = u_0(\underline{x}) & \boldsymbol{x} \in \Omega \end{array}$$

where $\kappa > 0$ is constant and u_0 is a given function. Suppose that *energy* for this system is defined as

$$E(u) := \frac{1}{2} \int_\Omega u^2 dV$$

where the single integral sign represents a multiple integral. Please show that E is decreasing. What is the maximum value of this energy?

$$\frac{dE}{dt} = \frac{1}{2}\int_\Omega \frac{\partial}{\partial t}(u^2)\,dV = \int_\Omega u u_t\,dV \overset{\text{PDE}}{=} \int_\Omega u(\kappa\Delta u)\,dV \overset{\text{Green's First Identity}}{=} \kappa\left(\underbrace{\int_{\partial\Omega} u u_n\,ds}_{\text{BC: } 0} - \int_\Omega |\nabla u|^2 dV\right)$$

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$$= \boxed{-\kappa \int_\Omega |\nabla u|^2 dV \le 0} \quad (+5)$$

- The maximum value of this energy is $E(0) = \frac{1}{2}\int_\Omega |u_0|^2 dV$ (+5)

- Notice that $\frac{dE}{dt} < 0$ unless u is constant which means $u \equiv 0$ given the BC.

2. (25 points) For $\boldsymbol{x} \in \mathbb{R}^3$, please give the solution (in symbols) of the following wave problem:

$$\text{PDE:} \qquad \Box u = 0 \qquad \boldsymbol{x} \in \mathbb{R}^3,\ t > 0$$

$$\text{IC:} \quad \begin{array}{ll} u(x,0) \equiv 0 & \boldsymbol{x} \in \mathbb{R}^3 \\ u_t(x,0) = 1 & |\boldsymbol{x}| = 1 \\ u_t(x,0) = 0 & |\boldsymbol{x}| \neq 1 \end{array}$$

So if u represents light, the initial conditions are zero everywhere except on the spherical shell with radius 1 centered at the origin where at time $t = 0$ there is a flash (nonzero time derivative of u). Please describe in words what the solution looks like.

This is a really interesting question. One answer is to note that $u(x,0) \equiv 0$ and $u_t(x,0) = 0$ a.e., so $u(x,t) = 0$ a.e. for $x \in \mathbb{R}^3$ and $t \geq 0$. But this is probably not the best solution. Another approach is to note that by (3), §9.2 (p. 234) Strauss

$$u(x,t) = \frac{1}{4\pi t} \iint_{|x-y|=t} \chi_{S_1(0)}(y)\, dS$$

where $c=1$, $\phi \equiv 0$ and $S_1(0)$ denotes the spherical shell with radius 1 centred at the origin. Thus for $x=0$ and $t=1$, the integral yields the surface area of $S_1(0)$, so $u(0,1)=1$. But for all other values of x and t, the domain of integration intersects $S_1(0)$ only on a curve (circle) or at a single point. Since the area under a circle or a point is zero, $u(x,t)=0$ unless $x=0$ and $t=1$.