MA4473 Name: *Solutions*

Final, Part 2 B Term, 2016

Show all work needed to reach your answers.

1. (20 points) The equation $u_{tt} - c^2\triangle u + m^2\sin(u) = 0$ is called the *Sine-Gordon equation*. What is the energy? Is it constant (prove or disprove)?

2. (20 points) Please describe and solve the first order PDE $u_x + u_y = u$ subject to the auxiliary condition $u(x, 0) = \cos x$. (Hint: you can get substantial credit for describing the solution even if you can not find it in closed form. A sketch of the x, y-plane may help.)

3. (20 points) Please find the harmonic function u on the disk $\Omega \subset \mathbb{R}^2$ with radius $r = 1$ which satisfies $u = 1 + \cos\theta$ on $\partial\Omega$. Is this function u unique (prove or disprove)?

1. For smooth solutions, if $u_{tt} - c^2\Delta u + m^2\sin(u) = 0$, then

$$0 = (u_{tt} - c^2\Delta u + m^2\sin(u))\,u_t = \frac{\partial}{\partial t}\left(u_t^2 + c^2|\nabla u|^2 - m^2\cos(u)\right) - c^2\operatorname{div}(u_t\nabla u)$$

Integrating over all of $\mathbb{R}^n$ and assuming that u goes to zero far from the origin, one finds that $E(t) := \iiint_{\mathbb{R}^n} u_t^2 + c^2|\nabla u|^2 - m^2\cos(u)\,dV$ satisfied

$$E'(t) = \iiint_{\mathbb{R}^n} \frac{\partial}{\partial t}\left(u_t^2 + c^2|\nabla u|^2 - m^2\cos(u)\right)dV = \iiint_{\mathbb{R}^n} c^2\operatorname{div}(u_t\nabla u)\,dV = 0.$$

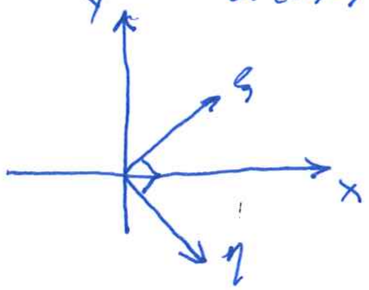
2. Since $\frac{\partial u}{\partial \xi} = \frac{\partial x}{\partial \xi}u_x + \frac{\partial y}{\partial \xi}u_y = u$ iff $\frac{\partial x}{\partial \xi} = 1$ and $\frac{\partial y}{\partial \xi} = 1$, the characteristics require $x = \xi + f(\eta)$ and $y = \xi + g(\eta)$. Since the ξ-direction relative to the (x,y)-plane is $\langle 1, 1\rangle$, the perpendicular η-direction is $\langle -1, 1\rangle$, the simplest choice is $\xi = \frac{x+y}{2}$ and $\eta = \frac{x-y}{2}$.

Now the solution of $\frac{\partial u}{\partial \xi} = u$ is $u = A(\eta)\,e^{\xi} = A\left(\frac{x-y}{2}\right)e^{\frac{x+y}{2}}$

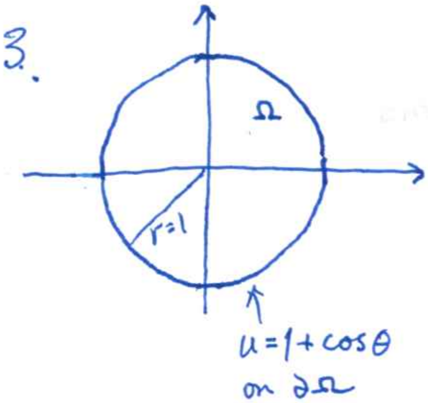
Because $u(x,0) = \cos(x) = A\left(\frac{x}{2}\right)e^{\frac{x}{2}}$, then $A\left(\frac{x}{2}\right) = e^{-\frac{x}{2}}\cos(x)$, or $A(s) = e^{-s}\cos(2s)$

and

$$u(x,t) = e^{\frac{y-x}{2}}\cos(x-y)\,e^{\frac{x+y}{2}} = \underline{e^{y}\cos(x-y)}$$



3.



Suppose $\exists$ u, v, two harmonic functions on Ω satisfying $u = v = 1+\cos\theta$ on $\partial\Omega$. But then $w = u - v$ is harmonic and $w = 0$ on $\partial\Omega$. By the maximum principle, $w \equiv 0$ on Ω. So $u = v \Rightarrow$ the solution must be unique.

What's the solution? Notice that since $\partial\Omega = \{(x,y) \mid r = 1\}$, on this boundary $u = 1 + \cos\theta = 1 + r\cos\theta = 1 + x$. Since $u(x,y) \equiv 1 + x$ is harmonic, this must be the unique solution.