MA500 Name: Solutions

Midterm (closed book, closed notes, no internet, no discussion) Fall 2014

I affirm that I have not consulted my text, notes or any reference, paper or electronic, or any person once I opened the envelope and began this midterm exam.

Signature: ______________________

Show all work needed to reach your answers. You may use any result we discussed in class, but cite by name any named result you use.

+15

1. (15 points) Suppose that $f : X \to Y$ and that A and B are subsets of X. Please show that $f(A \cap B) \subset f(A) \cap f(B)$. || Must start here.

Suppose that $y \in f(A \cap B)$. Then $\exists\, x \in A \cap B$ s.t. $y = f(x)$. So $x \in A$ and $x \in B$, implying that $f(x) = y \in f(A)$ and $f(x) = y \in f(B)$. Thus $y \in f(A) \cap f(B)$ and $f(A \cap B) \subset f(A) \cap f(B)$.

|| Must finish here.

+15

2. (15 points) Suppose $S \subset \mathbb{R}$ is a nonempty set that is bounded below. Can S have more than one greatest lower bound? Please explain in detail why or why not.

Suppose that x_1 and x_2 are both $glb(S)$. Then since x_1 is a $glb(S)$ and x_2 is a lower bound, $x_2 \leq x_1$. But since x_2 is a $glb(S)$ and x_1 is a lower bound, $x_1 \leq x_2$. Thus $x_1 = x_2$. So $x_1 = x_2 = glb(S)$ is unique.

3. (20 points) Consider the sequence of real numbers

$$\frac{1}{4}, \quad \frac{1}{4+\frac{1}{4}}, \quad \frac{1}{4+\frac{1}{4+\frac{1}{4}}}, \quad \ldots$$

Please show that this sequence is convergent and find its limit.

Suppose $\{a_n\}$ is this sequence; then $a_{n+1} = \frac{1}{4+a_n}$. First assume that the limit exists, i.e., $a_n \to a$. Then $a = \frac{1}{4+a} \Leftrightarrow a^2+4a-1=0 \Leftrightarrow a = \sqrt{5}-2$ (since $a>0$). So $\sqrt{5}-2$ is the only possible limit. +5

Now we must show that the limit exists. Consider +3 two subsequences $\{b_n\}$ and $\{c_n\}$ where the first is all of the odd elements of the original sequence, and the second is all of the even elements. So $b_1 = \frac{1}{4}$, and in general $b_{n+1} = \frac{1}{4+\frac{1}{4+b_n}} \Rightarrow b_{n+1} < b_1 \ \forall n \in \mathbb{Z}^+$. Now assume that $b_n < b_{n-1}$, and notice that $b_{n+1} = \frac{1}{4+\frac{1}{4+b_n}} < \frac{1}{4+\frac{1}{4+b_{n-1}}} = b_n$. So by induction, $\{b_n\}$ is a decreasing +2 sequence which is bounded +1 below (by zero), and thus it must converge +2: $\exists b \in \mathbb{R}$ s.t. $b_n \to b$. In addition, we have that $b = \frac{1}{4+\frac{1}{4+b}} = \frac{4+b}{4(4+b)+1} \Leftrightarrow b^2+4b-1=0 \Leftrightarrow b=\sqrt{5}-2$. +1

Next for $\{c_n\}$, we have that $c_1 = \frac{1}{4+1/4} = \frac{4}{17}$, and again $c_{n+1} = \frac{1}{4+\frac{1}{4+c_n}} \Rightarrow c_1 < c_{n+1} \ \forall n \in \mathbb{Z}^+$. So assume that $c_{n-1} < c_n$, and notice that $c_n = \frac{1}{4+\frac{1}{4+c_{n-1}}} < \frac{1}{4+\frac{1}{4+c_n}} = c_{n+1}$. Thus $\{c_n\}$ is an increasing +2 sequence which is bounded +1 above (by 1/4), implying that $c_n \to c$ +2 for some $c \in \mathbb{R}$. But as before, c must satisfy $c = \frac{1}{4+\frac{1}{4+c}} \Leftrightarrow c^2+4c-1=0 \Leftrightarrow c=\sqrt{5}-2$. +1

Since both subsequences converge to the same value, the entire sequence $\{a_n\}$ must converge to this value.

4. (20 points) Let (E, d) be any metric space. If $F \subset E$, then F is closed iff F contains its cluster points (limit points).

($\Rightarrow$) First suppose that F is closed. Then $\complement F$ is open.

+10 Suppose x is a cluster point of F and that $x \notin F$. So $x \in \complement F$, and since $\complement F$ is open, $\exists\, \epsilon > 0$ such that $B_\epsilon(x) \subset \complement F$. But this contradictions the defintion of x being a cluster point ($B_\epsilon(x)$ must contain an infinite number points of F). So $x \in F$, and any cluster point of F must be contained in F.

($\Leftarrow$) Now suppose F contains its cluster points. If F is not

+10 closed, then $\complement F$ is not open. So $\exists\, x \in \complement F$ s.t. $\forall \epsilon > 0$, $B_\epsilon(x) \not\subset \complement F$. Hence $\forall n \in \mathbb{Z}^+$, $\exists\, x_n \in B_{1/n}(x) \cap F$, implying that x is a cluster point of F. Thus if F contains its cluster points, F must be closed.

5. (15 points) Let (E, d_0) be a metric space, and for any $x, y \in E$, define

$$d(x,y) := 2d_0(x,y).$$

Please prove or disprove in detail that (E, d) is also a metric space.

To show that (E,d) is a metric space, one must verify four conditions, all based on (E,d_0) being a metric space:

+3 1) $\forall x,y \in E$; $d(x,y) = 2d_0(x,y) \geq 0$.

+4 2) If $x=y$, then $d(x,y) = 2d_0(x,y) = 0$. If $d(x,y)=0$, then $d_0(x,y)=0 \Rightarrow x=y$.

+3 3) $d(x,y) = 2d_0(x,y) = 2d_0(y,x) = d(y,x)$

+5 4) Since $\forall x,y,z \in E$ $d_0(x,y) \leq d_0(x,z) + d_0(z,y)$, one has that $2d_0(x,y) \leq 2d_0(x,z) + 2d_0(z,y)$ or

$$d(x,y) \leq d(x,z) + d(z,y)$$

This establishes the Δ-inequality.

Note: No where is it given that $d_0(x,y) = |x-y|$

6. (15 points) Consider the set $S = \{0, 1\}$ along with the operations $+$ and $\times$ defined by the tables below. With these operations, S is a field. Can this field be ordered? Please explain in detail why or why not.

$+$	0	1
0	0	1
1	1	0

$\times$	0	1
0	0	0
1	0	1

Suppose this field is ordered. Then either $0<1$ or $1<0$ ($0 \neq 1$ in any field). If $0<1$, then $0+1 < 1+1 \Leftrightarrow 1 < 0$ $\rightarrow\leftarrow$. On the other hand, if $1<0$, then $1+1 < 0+1 \Leftrightarrow 0<1$ which is again a contradiction. So this field can not be ordered.

+10: For considering only $0<1$.