Evolution of a Bubble in Extensional Flow

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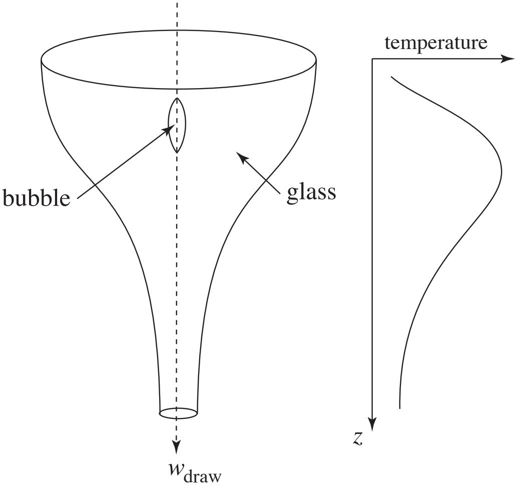
MPI 2008 at WPI

June 20, 2008

The Problem



Questions

- What shape is the final bubble?
  - Fore/Aft symmetric?
  - Same as uniform stretching?
  - Longer/shorter than expected?
- What controls final bubble shape?
  - Temperature profile
  - Glass viscosity
  - Surface tension
  - Gas pressure
  - Initial bubble shape
- Are previous analyses of bubbles in shear flow relevant?

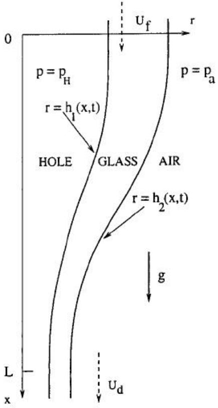
Model Assumptions

- Newtonian viscous flow
- Variable viscosity due to temperature (prescribed)
- Constant surface tension
- Axisymmetric flow
- Ideal gas with spatially uniform properties in the bubble

Previous Work

"The Mathematical Modelling of Capillary Drawing for Holey Fibre Manufacture," Fitt, et.al.

- Optical fibers that contain air holes, inner and outer radii of air hole is taken into consideration
- Gave three governing equations:
  - Conservation of mass
  - Normal stress condition on each surface
- Our model is obtained by making the inner radius much less than the outer radius



Equations

Glass Behavior

$$\frac{\partial}{\partial z}\left[wR^2\right] = 0$$

$$R^2 \,\mathrm{Re}\left[w\frac{\partial w}{\partial z}\right] = \frac{\partial}{\partial z}\left[3R^2\mu(T(z))\frac{\partial w}{\partial z} + \sigma R\right]$$

Bubble Behavior

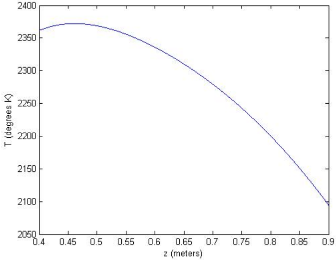
$$\frac{\partial h^2}{\partial t} + \frac{\partial}{\partial z}\left[h^2 w\right] = \frac{1}{\mu(T(z))}\left[h^2 P - \sigma h\right]$$

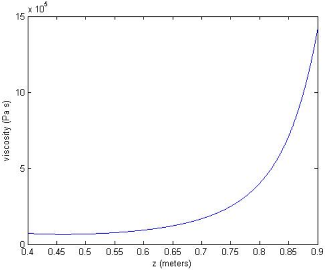
$$\frac{P}{R_g}\int_{z_{\mathrm{min}}}^{z_{\mathrm{max}}} \frac{\pi h^2(z,t)}{T(z)}dz = M.$$

Characteristic Scales

Parameter		Range
Length		$1\ m$
Glass Radius	$R(z)$	$10^{-1} - 10^{-4}\ m$
Bubble Radius	$h(z,t)$	$10^{-5} - 10^{-8}\ m$
Viscosity	$\mu(T(z))$	$10^{5} - 10^{6}\ kgm^{-1}s^{-1}$
Axial Velocity	$w(z)$	$10^{-3} - 10^{1}\ ms^{-1}$
Surface Tension	σ	$.25\ Nm^{-1}$

Temperature and Viscosity profiles





Data fit

Question

At steady state, do the equations which govern the velocity w result in a velocity profile which fits the provided data?

We try to answer this question using numerical methods, by focusing on the first two equations.

Simplest approximation

Assume Reynolds number, σ is small.

Equations

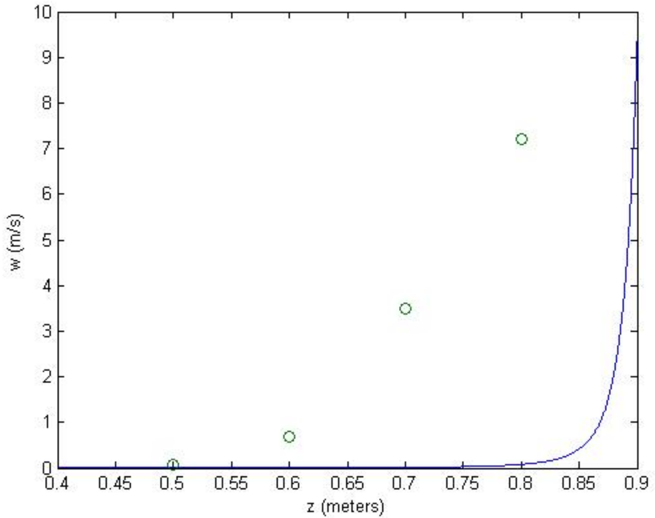
$$\frac{\partial}{\partial z}\left[wR^2\right] = 0$$

$$\frac{\partial}{\partial z}\left[3R^2\mu(T(z))\frac{\partial w}{\partial z}\right] = 0$$

Method

- First equation $\Rightarrow R^2 w = Q$ (constant)
- Set $R^2 = Q/w \Rightarrow$ Second equation is DE in terms of w
- Integrate twice to get expression for w
- Apply BCs numerically and solve for w numerically

Basic approximation



Second approximation

Neglect only inertia

Equations

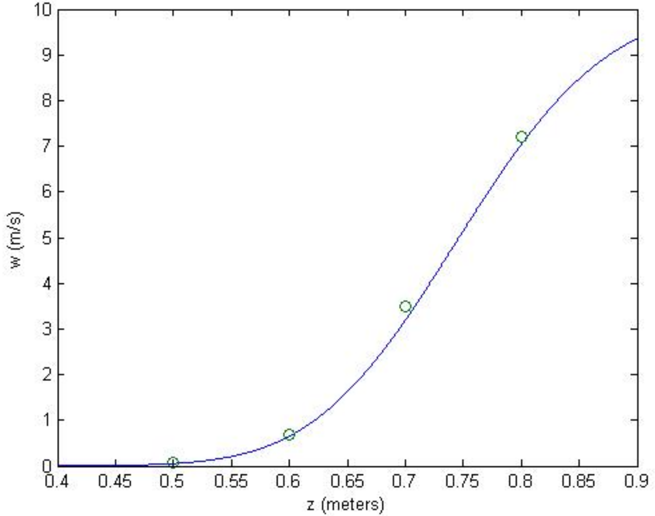
$$\frac{\partial}{\partial z}\left[wR^2\right] = 0$$

$$\frac{\partial}{\partial z}\left[3R^2\mu(T(z))\frac{\partial w}{\partial z} - \sigma R\right] = 0$$

Method

- Equation (1) $\Rightarrow R^2 = Q$ (constant)
- Set $R^2 = Q/w \Rightarrow$ Second equation is DE in terms of w
- Integrate once, then apply a shooting method on constant of integration (binary search and forward Euler) to find w

Second approximation



Third approximation

Surface tension and inertia effects are included.
Rewrite the second equation with $R^2 = Q/w$:

Equations

$$\frac{\partial}{\partial z}\left[wR^2\right] = 0$$

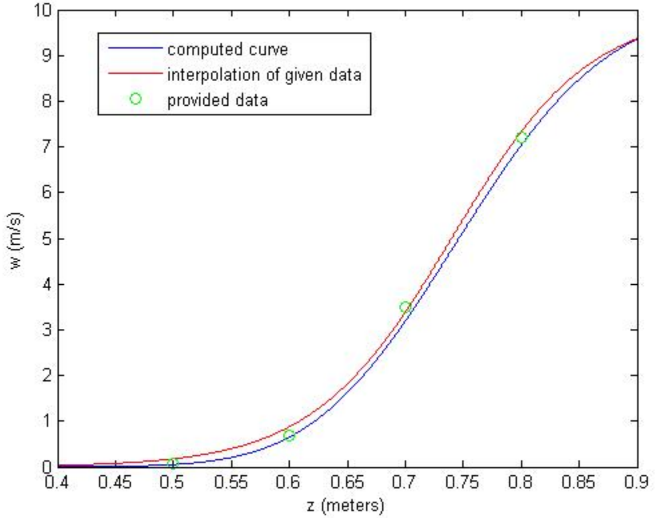
$$\frac{\partial}{\partial z}\left[3\frac{Q}{w}\mu(T(z))\frac{\partial w}{\partial z} + \sigma\sqrt{\frac{Q}{w}} - Q\,Re\,w\right] = 0$$

Follow same procedure as with Second approximation

Results are similar

Comparison of Fits

A tanh fit to the data was used for other work on this problem



Bubble Dynamics

Having found a good fit for w and R it is now possible to study the bubble behavior equations.

Governing Equations

$$\frac{\partial h^2}{\partial t} + \frac{\partial}{\partial z}\left[h^2 w\right] = \frac{1}{\mu(T(z))}\left[h^2 P - \sigma h\right]$$

$$\int_{z_{\min}}^{z_{\max}} \rho_b(z,t)\pi h^2(z,t)dz = M, \qquad P(t) = \rho_b(z,t)R_g T(z)$$

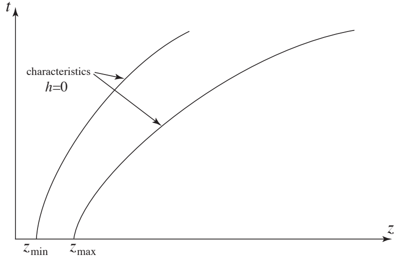
- $w(z)$ given from previous analysis
- initial conditions: $h(z,0)$ for $z_{\min} \leq z \leq z_{\max}$ and $P(0)$ given

Analysis

- Pinch-off behavior
- Numerical solution

Pinch-off Analysis

General solution behavior



- $h(z,t)$ evolves on characteristics starting between z_{min} and z_{max}
- If no pinch-off occurs, bubble stretches according to limiting characteristics
- Bubble may pinch off, so bubble may be shorter than expected

Pinch-off model

Assumptions

- P, μ and σ taken to be constant
- Velocity profile taken to be $w(z) = \gamma z$ for a fixed γ.

Equation to solve:

$$\frac{\partial h^2}{\partial t} + \gamma z \frac{\partial h^2}{\partial z} = \left(\frac{P}{\mu} - \gamma \right) h^2 - \frac{\sigma}{\mu} h$$

subject to $h(z, 0)$ given.
Solve using method of characteristics

Solution

Along characteristics,

$$h(t) = \left[\frac{(\alpha h_0 - \beta)\exp\left(\frac{\alpha t}{2}\right) + \beta}{\alpha}\right],$$

while $h > 0$, otherwise $h = 0$, where

$$\alpha = \frac{P}{\mu} - \gamma, \qquad \beta = \frac{\sigma}{\mu}.$$

Result:

- For $P > \gamma\mu$, no pinch off occurs provided $h_0 > \frac{\beta}{\alpha}$.
- For $P < \gamma\mu$, the bubble pinches off at

$$\tilde{t} = \frac{2}{-\alpha} \ln\left(\frac{\beta + (-\alpha)h_0}{\beta}\right),$$

for any positive value of h_0.

Numerical Solution of Bubble Dynamics

Characteristic form:

$$\frac{dh^2}{dt} = -w_z h^2 + \frac{1}{\mu(T(z))}\left[Ph^2 - \sigma h\right], \qquad \text{on } \frac{dz}{dt} = w(z)$$

Fixed mass constraint:

$$\frac{P}{R_g}\int_{z_{\min}}^{z_{\max}} \frac{\pi h^2(z,t)}{T(z)}\, dz = M$$

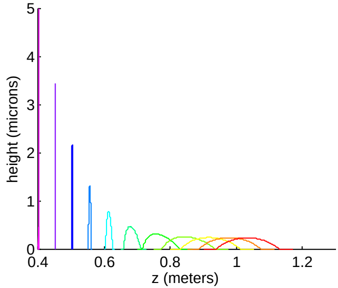
Basic numerical scheme (assuming $w(z)$ is known):

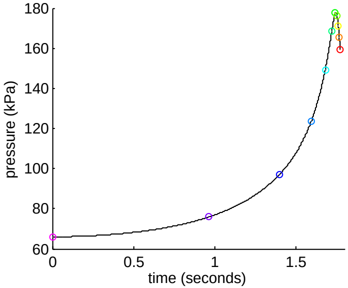
- Pick mesh points $\{z_j(0)\} \in [z_{\min}, z_{\max}]$, specify $h_j(0)$ (ICs).
- Advance $z_j(t)$ and $h_j(t)$ to $t + \Delta t$ using the characteristic form with $P(t)$ held fixed.
- Compute $P(t + \Delta t)$ using fixed mass constraint.

Results

Initial state:

- Ellipsoidal Bubble: 1mm length, 5μm radius (maximum)
- Bubble pressure balanced by surface tension





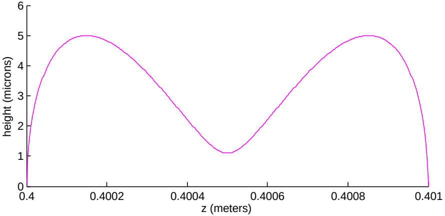
Observations

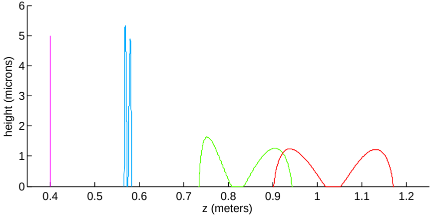
- Bubble radius shrinks due to stretching
- Smaller bubble radius requires higher pressure to balance surface tension
- Bubble shrinks more to raise pressure
- Pressure reduction occurs when stretching ceases and temperature continues to drop

Further Results

Initial state:

- Dumb-bell shaped Bubble: 1mm length, 5μm radius (maximum), 1μm radius (minimum near center)
- Bubble pressure balanced by surface tension





Observation

Bubbles may pinch off in the middle and break in two

Model 2: "Leaky bubble"

Extensions to previous model:

- Gas may diffuse from bubble into surrounding glass (locally)
- Bubble mass no longer conserved
- Gas flux due to diffusion and bubble elongation

Model Equations:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial z}(w\rho) + \frac{\partial}{\partial r}(u\rho) = \frac{D}{r}\frac{\partial}{\partial r}(r\frac{\partial \rho}{\partial r})$$

where $\rho(z,t)$ is density of gas in glass, $h(z,t)$ is bubble radius, and $u = \frac{h}{r}\frac{\partial h}{\partial z}$.

Initial data for ρ specified, and the boundary condition ρ at $r = h$ determined from pressure in bubble

Model 2: "Leaky bubble"

Gas conservation of bubble ($\rho_b =$ bubble density)

$$\int_{z_{\min}}^{z_{\max}} \rho_b \pi h^2 \, dz = M, \quad \frac{dM}{dt} = \int_{z_{\min}}^{z_{\max}} D \left. \frac{\partial \rho}{\partial r} \right|_{r=h} 2\pi h \, dz.$$

Pressure inside bubble related to density by gas law

$$P(t) = \rho_b(z,t) R_g T(z)$$

and $\rho = \Lambda \rho_b$ on $r = h$, Λ is the solubility coefficient. This must be coupled to original three equations for w, R, h.

To be done!

Summary & future directions

- Formulated a model for finite length axisymmetric bubble in glass fibre undergoing extensional flow (c.f. Fitt et al. "holey fibre" model)
- Four coupled equations for axial velocity $w(z,t)$, fibre radius $R(z,t)$, bubble radius $h(z,t)$, and bubble pressure $P(t)$. Furnace temperature $T(z)$ (and hence viscosity) specified; mass of gas in bubble fixed.
- Model predicts that bubble cannot extend beyond limiting characteristics.
- Pinch-off model demonstrates that finite-time "pinch-off" of bubbles can occur, causing shorter bubbles
- Numerical code developed to solve governing equations for a fixed mass bubble
- Further numerical investigation of model is needed to investigate other possible phenomena under different drawing conditions and with different parameter values
- Formulated a model in which gas may leak from the bubble into surrounding glass – this model has yet to be studied