Evolution of a Bubble in Extensional Flow

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The Problem

- bubble
- glass bubble
- temperature
- draw
- $w_{\text{draw}}$
- $z$

Evolution of a Bubble in Extensional Flow
Questions

- What shape is the final bubble?
  - Fore/Aft symmetric?
  - Same as uniform stretching?
  - Longer/shorter than expected?

- What controls final bubble shape?
  - Temperature profile
  - Glass viscosity
  - Surface tension
  - Gas pressure
  - Initial bubble shape

- Are previous analyses of bubbles in shear flow relevant?
Model Assumptions

- Newtonian viscous flow
- Variable viscosity due to temperature (prescribed)
- Constant surface tension
- Axisymmetric flow
- Ideal gas with spatially uniform properties in the bubble
Previous Work


- Optical fibers that contain air holes, inner and outer radii of air hole is taken into consideration
- Gave three governing equations:
  - Conservation of mass
  - Normal stress condition on each surface
- Our model is obtained by making the inner radius much less than the outer radius
Glass Behavior

\[ \frac{\partial}{\partial z} \left[ wR^2 \right] = 0 \]

\[ R^2 \text{Re} \left[ w \frac{\partial w}{\partial z} \right] = \frac{\partial}{\partial z} \left[ 3R^2 \mu(T(z)) \frac{\partial w}{\partial z} + \sigma R \right] \]

Bubble Behavior

\[ \frac{\partial h^2}{\partial t} + \frac{\partial}{\partial z} \left[ h^2 w \right] = \frac{1}{\mu(T(z))} \left[ h^2 P - \sigma h \right] \]

\[ \frac{P}{R_g} \int_{z_{\text{min}}}^{z_{\text{max}}} \frac{\pi h^2(z,t)}{T(z)} \, dz = M. \]
## Characteristic Scales

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length</td>
<td>$1 \text{ m}$</td>
</tr>
<tr>
<td>Glass Radius</td>
<td>$R(z)$, $10^{-1} - 10^{-4} \text{ m}$</td>
</tr>
<tr>
<td>Bubble Radius</td>
<td>$h(z, t)$, $10^{-5} - 10^{-8} \text{ m}$</td>
</tr>
<tr>
<td>Viscosity</td>
<td>$\mu(T(z))$, $10^5 - 10^6 \text{ kgm}^{-1}\text{s}^{-1}$</td>
</tr>
<tr>
<td>Axial Velocity</td>
<td>$w(z)$, $10^{-3} - 10^{1} \text{ m}\text{s}^{-1}$</td>
</tr>
<tr>
<td>Surface Tension</td>
<td>$\sigma$, $.25 \text{ Nm}^{-1}$</td>
</tr>
</tbody>
</table>
Temperature and Viscosity profiles

![Graph of Temperature Profile](image1)

![Graph of Viscosity Profile](image2)
Question
At steady state, do the equations which govern the velocity $w$ result in a velocity profile which fits the provided data?

We try to answer this question using numerical methods, by focusing on the first two equations.
Simplest approximation

Assume Reynolds number, $\sigma$ is small.

Equations

$$\frac{\partial}{\partial z} \left[ wR^2 \right] = 0$$

$$\frac{\partial}{\partial z} \left[ 3R^2 \mu(T(z)) \frac{\partial w}{\partial z} \right] = 0$$

Method

- First equation $\Rightarrow R^2 w = Q$ (constant)
- Set $R^2 = Q/w$ $\Rightarrow$ Second equation is DE in terms of $w$
- Integrate twice to get expression for $w$
- Apply BCs numerically and solve for $w$ numerically
Basic approximation
Second approximation

Neglect only inertia

**Equations**

\[
\frac{\partial}{\partial z} \left[ wR^2 \right] = 0
\]

\[
\frac{\partial}{\partial z} \left[ 3R^2 \mu(T(z)) \frac{\partial w}{\partial z} - \sigma R \right] = 0
\]

**Method**

- Equation (1) \( \Rightarrow R^2 = Q \) (constant)
- Set \( R^2 = Q/w \) \( \Rightarrow \) Second equation is DE in terms of \( w \)
- Integrate once, then apply a shooting method on constant of integration (binary search and forward Euler) to find \( w \)
Second approximation
Surface tension and inertia effects are included. Rewrite the second equation with $R^2 = Q/w$:

Equations

\[
\frac{\partial}{\partial z} \left[ wR^2 \right] = 0
\]

\[
\frac{\partial}{\partial z} \left[ 3 \frac{Q}{w} \mu(T(z)) \frac{\partial w}{\partial z} + \sigma \sqrt{\frac{Q}{w}} - Q \text{Re} w \right] = 0
\]

Follow same procedure as with Second approximation

Results are similar
Comparison of Fits

A tanh fit to the data was used for other work on this problem.
Having found a good fit for $w$ and $R$ it is now possible to study the bubble behavior equations.

**Governing Equations**

\[
\frac{\partial h^2}{\partial t} + \frac{\partial}{\partial z} \left[ h^2 w \right] = \frac{1}{\mu(T(z))} \left[ h^2 P - \sigma h \right]
\]

\[
\int_{z_{\text{min}}}^{z_{\text{max}}} \rho_b(z, t) \pi h^2(z, t) dz = M, \quad P(t) = \rho_b(z, t) R_g T(z)
\]

- $w(z)$ given from previous analysis
- initial conditions: $h(z, 0)$ for $z_{\text{min}} \leq z \leq z_{\text{max}}$ and $P(0)$ given

**Analysis**

- Pinch-off behavior
- Numerical solution
Pinch-off Analysis

General solution behavior

- \( h(z, t) \) evolves on characteristics starting between \( z_{\text{min}} \) and \( z_{\text{max}} \)
- If no pinch-off occurs, bubble stretches according to limiting characteristics
- Bubble may pinch off, so bubble may be shorter than expected
Pinch-off model

Assumptions

- $P$, $\mu$ and $\sigma$ taken to be constant
- Velocity profile taken to be $w(z) = \gamma z$ for a fixed $\gamma$.

Equation to solve:

$$\frac{\partial h^2}{\partial t} + \gamma z \frac{\partial h^2}{\partial z} = \left( \frac{P}{\mu} - \gamma \right) h^2 - \frac{\sigma}{\mu} h$$

subject to $h(z, 0)$ given.

Solve using method of characteristics
Along characteristics,

\[ h(t) = \left[ \frac{(\alpha h_0 - \beta) \exp(\frac{\alpha t}{2}) + \beta}{\alpha} \right], \]

while \( h > 0 \), otherwise \( h = 0 \), where

\[ \alpha = \frac{P}{\mu} - \gamma, \quad \beta = \frac{\sigma}{\mu}. \]

Result:

- For \( P > \gamma \mu \), no pinch off occurs provided \( h_0 > \frac{\beta}{\alpha} \).
- For \( P < \gamma \mu \), the bubble pinches off at

\[ \tilde{t} = \frac{2}{-\alpha} \ln \left( \frac{\beta + (-\alpha)h_0}{\beta} \right), \]

for any positive value of \( h_0 \).
Numerical Solution of Bubble Dynamics

Characteristic form:

\[
\frac{dh^2}{dt} = -w_z h^2 + \frac{1}{\mu(T(z))} \left[ Ph^2 - \sigma h \right], \quad \text{on} \quad \frac{dz}{dt} = w(z)
\]

Fixed mass constraint:

\[
\frac{P}{R_g} \int_{z_{\text{min}}}^{z_{\text{max}}} \frac{\pi h^2(z, t)}{T(z)} \, dz = M
\]

Basic numerical scheme (assuming \( w(z) \) is known):

- Pick mesh points \( \{z_j(0)\} \in [z_{\text{min}}, z_{\text{max}}] \), specify \( h_j(0) \) (ICs).
- Advance \( z_j(t) \) and \( h_j(t) \) to \( t + \Delta t \) using the characteristic form with \( P(t) \) held fixed.
- Compute \( P(t + \Delta t) \) using fixed mass constraint.
Results

Initial state:
- Ellipsoidal Bubble: 1mm length, 5µm radius (maximum)
- Bubble pressure balanced by surface tension

Observations
- Bubble radius shrinks due to stretching
- Smaller bubble radius requires higher pressure to balance surface tension
- Bubble shrinks more to raise pressure
- Pressure reduction occurs when stretching ceases and temperature continues to drop
Further Results

Initial state:
- Dumb-bell shaped Bubble: 1mm length, 5µm radius (maximum), 1µm radius (minimum near center)
- Bubble pressure balanced by surface tension

Observation
Bubbles may pinch off in the middle and break in two
Model 2: “Leaky bubble”

Extensions to previous model:
- Gas may diffuse from bubble into surrounding glass (locally)
- Bubble mass no longer conserved
- Gas flux due to diffusion and bubble elongation

Model Equations:

\[
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial z} (w \rho) + \frac{\partial}{\partial r} (u \rho) = \frac{D}{r} \frac{\partial}{\partial r} (r \frac{\partial \rho}{\partial r})
\]

where \( \rho(z, t) \) is density of gas in glass, \( h(z, t) \) is bubble radius, and \( u = \frac{h}{r} \frac{\partial h}{\partial z} \).

Initial data for \( \rho \) specified, and the boundary condition \( \rho \) at \( r = h \) determined from pressure in bubble.
Model 2: “Leaky bubble”

Gas conservation of bubble ($\rho_b = \text{bubble density}$)

$$\int_{z_{\text{min}}}^{z_{\text{max}}} \rho_b \pi h^2 \, dz = M, \quad \frac{dM}{dt} = \int_{z_{\text{min}}}^{z_{\text{max}}} D \frac{\partial \rho}{\partial r} \bigg|_{r=h} 2\pi h \, dz.$$  

Pressure inside bubble related to density by gas law

$$P(t) = \rho_b(z, t) R_g T(z)$$

and $\rho = \Lambda \rho_b$ on $r = h$, $\Lambda$ is the solubility coefficient. This must be coupled to original three equations for $w$, $R$, $h$.

To be done!
Summary & future directions

- Formulated a model for finite length axisymmetric bubble in glass fibre undergoing extensional flow (c.f. Fitt et al. “holey fibre” model)
- Four coupled equations for axial velocity $w(z, t)$, fibre radius $R(z, t)$, bubble radius $h(z, t)$, and bubble pressure $P(t)$. Furnace temperature $T(z)$ (and hence viscosity) specified; mass of gas in bubble fixed.
- Model predicts that bubble cannot extend beyond limiting characteristics.
- Pinch-off model demonstrates that finite-time “pinch-off” of bubbles can occur, causing shorter bubbles.
- Numerical code developed to solve governing equations for a fixed mass bubble.
- Further numerical investigation of model is needed to investigate other possible phenomena under different drawing conditions and with different parameter values.
- Formulated a model in which gas may leak from the bubble into surrounding glass – this model has yet to be studied.