TOPICS

- More Solving Equations... Approximately
  \rightarrow Newton’s Error
  \rightarrow Fixed Point Method
- Beginning Convergence
  \rightarrow Sequences of Numbers

Recall Newton... Picture:
Newton’s Method

\[ x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \]

Question:

How fast does the method converge?

Example:

Approximating \( \sqrt{2} \) by solving \( x^2 - 2 = 0 \)

You double the number of correct digits at each step.

<table>
<thead>
<tr>
<th>Step</th>
<th>( x_n )</th>
<th>Correct Digits</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1.183</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1.4621</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>1.4149984</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>1.4142137800</td>
<td>7</td>
</tr>
<tr>
<td>( \infty )</td>
<td>1.4142135623</td>
<td>**</td>
</tr>
</tbody>
</table>
Remark:

Recursion builds a sequence

\[ x_0, x_1, x_2, \ldots x_n, x_{n+1}, \ldots \]

and you hope that

\[ \lim_{n \to \infty} x_n = \sqrt{2} \]

Theorem:

Newton is always this fast.

\[ |x_{n+1} - r| \leq \frac{M}{2m} (x_n - r)^2 \]

where

\[ f'(x) \geq m > 0 \quad \text{and} \quad |f''(x)| \leq M \]

and \( r \) is a root for \( f(x) = 0 \).
Justification: Taylor!

\[ 0 = f(r) = f(x_n) + f'(x_n)(r - x_n) + \frac{f''(c)}{2} \cdot (r - x_n)^2 \]

Rearrange terms:

\[ x_n - \frac{f(x_n)}{f'(x_n)} - r = \frac{f''(c)}{2f'(x_n)}(x_n - r)^2 \]

Fixed Point Methods

Example: You can solve \( \cos(\theta) = \theta \) with one finger

1. Put your calculator in radian mode
2. Start anywhere (call it \( \theta_0 \))
3. Hit the cos key until you get tired
**WHY Does it Work?**

You are building a sequence

$$\theta_0, \theta_1, \theta_2, \ldots, \theta_n, \ldots$$

and you hope that

$$\theta^* = \lim_{n \to \infty} \theta_n = \cos(\theta^*)$$

Awfully Optimistic!

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**Aside:**

IF there is a limit, it has to be a fixed point:

$$\theta_{n+1} = \cos(\theta_n)$$

$$\downarrow \quad \downarrow$$

$$\theta^* = \cos(\theta^*)$$
Here is Why:

Look at the difference:

\[
\begin{align*}
\theta_{n+1} - \theta_n &= f(\theta_n) - f(\theta_{n-1}) \\
&= f'(\zeta) \cdot (\theta_n - \theta_{n-1}) \\
&< p \cdot (\theta_n - \theta_{n-1})
\end{align*}
\]

IF the derivative is always less than \( p < 1 \).

Error always decreases: This is the Key!

Quick Summary:

1. Newton and Bisection will solve \( f(x) = 0 \)
2. Fixed point method will solve \( f(x) = x \)
3. All methods are recursive (and easy to program).
New Stuff:

How do you PROVE that

\[ \lim_{n \to \infty} \frac{1}{n^2} = 0 \, ? \]

Intuition:

When \( n \) is big, \( 1/n^2 \) is small.

Mathematics:

For any number \( \epsilon > 0 \), there exists a \( N > 0 \) such that

\[ n > N \quad \text{guarantees} \quad \left| \frac{1}{n^2} - 0 \right| < \epsilon \]

Definition of Convergence

Say that

\[ \lim_{n \to \infty} a_n = L \]

if for any number \( \epsilon > 0 \), there exists a \( N > 0 \) such that

\[ |a_n - L| < \epsilon \quad \text{whenever} \quad n > N. \]
Apply the Definition

\[ \lim_{n \to \infty} \frac{1}{n^2} = 0 \]

Find a formula for \( N \) in terms of \( \epsilon \).

Example:

Does \( a_n = 2 + (0.99)^n \) converge?
Two Key Limits:

\#1: \( \lim_{n \to \infty} \frac{1}{n} = 0 \)

\#2: \( \lim_{n \to \infty} p^n = 0 \) if and only if \(-1 < p < 1\)

Many Key Theorems
ANNOUNCEMENTS:

● Homework #4 Due Tomorrow in Lecture
  Sec. 10.3: 7, 10, 13, 17;
  Sec. 10.4: 8, 15, 18, 19

● Quiz #4: Tomorrow in Lecture: Solving Equations

● Make-up for ALL Quizzes: Friday (Sign up Wednesday)

  2:00,  2:30,  3:00,  3:30

  in Stratton Hall 106