1. Find the general solutions of the following differential equations. Where initial conditions are given also solve the particular initial value problem.

(a) \( y'' - 6y' + 9y = 0 \)

\[ \text{char. eq: } r^2 - 6r + 9 = 0 \text{ or } (r - 3)^2 = 0 \]

\[ r = 3 \text{, double root so } \]

\[ y = c_1 e^{3x} + c_2 xe^{3x} \]

(b) \( y'' - 5y' + 6y = 0 \) with \( y(0) = 1 \), \( y'(0) = 2 \)

\[ \text{char. eq: } r^2 - 5r + 6 = 0 \text{ or } (r - 3)(r - 2) = 0 \]

\[ r_1 = 3, r_2 = 2 \text{ so } \]

\[ y = c_1 e^{3x} + c_2 e^{2x} \]

\[ y' = 3c_1 e^{3x} + 2c_2 e^{2x} \]

Apply \( y' IC \)

\[ y(0) = c_1 + c_2 = 1 \]

\[ y'(0) = 3c_1 + 2c_2 = 2 \]

\[ c_1 = 1 - c_2 \text{, sub into 2nd eq. } \]

\[ 3 - 3c_2 + 2c_2 = 2 \text{ or } -c_2 = -1, c_2 = 1 \]

\[ c_1 = 1 - c_2 = 0 \text{ so } \]

\[ y = e^{2x} \]
2. Answer the following questions about linear, first order differential equations.

(a) The general form for a first-order, linear differential equation is \( y' + p(x)y = q(x) \). What conditions on \( p(x) \) and \( q(x) \) guarantee that solutions to IVPs will exist and be unique? Explain by using our basic existence and uniqueness result.

\[
\begin{align*}
\text{Write as } & \quad y' = -p(x)y + q(x) \quad \text{so } F(x, y) = -p(x)y + q(x) \\
F_y(x, y) &= -p(x). \text{ If } p(x) \text{ and } q(x) \text{ are continuous, then our existence and uniqueness theorem says that solutions to IVP will exist and be unique.}
\end{align*}
\]

(b) Suppose \( y(x) \) is a solution to the IVP \( y' + p(x)y = q(x), \, y(0) = 1 \) where \( q(x) \) is a continuous function and \( p(x) = \frac{1}{(x+2)} \). Would you expect \( x = -2 \) to be contained in the interval of existence of \( y(x) \)? Why or why not?

\( p(x) \text{ is continuous except at } x = -2, \text{ so solution to IVP will exist on an interval containing } x = 0. \text{ Would not expect interval to contain } x = -2 \text{ since } p(x) \text{ is not continuous there.} \)
3. Consider the following differential equation.

\[ y' = \frac{xe^x}{y^2 - 3} \]

Describe the points of the plane where our basic existence and uniqueness result does not guarantee a unique solution to an IVP. Do not solve the differential equation.

\[ F(x, y) = \frac{xe^x}{y^2 - 3} \]

\[ F_y(x, y) = -\frac{xe^x}{(y^2 - 3)^2} \cdot 2y \]

\[ F \text{ and } F_y \text{ are continuous except when } y^2 = 3 \text{ or } y = \pm \sqrt{3}, \text{ so our existence and uniqueness theorem guarantees a unique solution to any IVP } y' = \frac{xe^x}{y^2 - 3}, y(a) = b \text{ will exist in an interval } (a-h, a+h) \text{ for some } h > 0 \text{ as long as } b \neq \sqrt{3} \text{ or } b \neq -\sqrt{3}. \]
4. Consider the following model for a population $P(t)$. The constants $N$ and $k$ that appear in the model are both positive.

$$ \frac{dP}{dt} = k(P - N) $$

(a) Solve the differential equation given above using separation of variables.

$$ \frac{dP}{P - N} = k \, dt \quad \text{(lost solution } P = N) $$

Integrate, get

$$ \ln |P - N| = kt + c_i $$

or

$$ |P - N| = e^{kt + c_i} = e^{c_i} e^{kt} $$

As usual, replace $e^{c_i}$ with $C$, drop absolute value to obtain

$$ P - N = Ce^{kt} $$

(b) Describe the behavior of solutions of this model. Your explanation should include a graph with representative solutions.

Since $k > 0$, get exponential growth for any IC with $P(0) \neq N$.
5. Consider the following differential equation.

\[ y' + y = (x + 2)e^{-x} \]

(a) Find the general solution to this differential equation.

\[ p(x) = 1, \quad \text{so} \quad \int p(x) \, dx = x \quad \text{and} \quad u(x) = e^x, \quad \text{so} \quad \text{DE becomes} \]

\[ (e^x y)' = x + 2, \quad \text{integrate, get} \quad e^x y \]

\[ e^x y = \frac{x^2}{2} + 2x + C \]

or

\[ y = \left( \frac{x^2}{2} + 2x \right)e^{-x} + Ce^{-x} \]

(b) Use your general solution to solve the IVP consisting of the differential equation above and the initial condition \( y(0) = 3 \).

\[ y(0) = C + 3 \quad \text{so} \quad \text{solution is} \]

\[ y = \left( \frac{x^2}{2} + 2x \right)e^{-x} + 3e^{-x} \]
6. Consider the following DE.
\[ y'' + 4y' + 8y = q(x) \]

(a) Find a general solution of the homogeneous equation.
\[ \text{char. eq } r^2 + 4r + 8 = 0 \text{ or } (r+2)^2 + 4 = 0 \]
\[ \text{roots } r = -2 \pm 2i \]
\[ Y_h = c_1 e^{-2x} \cos(2x) + c_2 e^{-2x} \sin(2x) \]

(b) If \( q(x) = xe^{-2x} \sin(2x) \), write down the form for the particular solution. Do not solve for the coefficients.
\[ \text{Try } \quad Y_p = (A_0 + A_1x)e^{-2x} \cos(2x) + (B_0 + B_1x)e^{-2x} \sin(2x) \]
\[ \text{won't work, so} \]
\[ \text{Try } \quad Y_p = x(A_0 + A_1x)e^{-2x} \cos(2x) + x(B_0 + B_1x)e^{-2x} \sin(2x) \]
\[ \text{will work.} \]

(c) If \( q(x) = 12e^{-2x} \), find the particular solution.
\[ \text{Try } \quad Y = Ae^{-2x} \text{ will work. } \quad Y_p' = -2Ae^{-2x} \]
\[ Y_p'' = 4Ae^{-2x} \text{ \&} \]
\[ Y_p'' + 4Y_p' + 8Y_p = 4Ae^{-2x} - 8Ae^{-2x} + 8Ae^{-2x} = 4Ae^{-2x} \]
\[ A \neq 0 \quad 4Ae^{-2x} = 12e^{-2x} \text{ so } A = 3 \]
\[ \text{and } \quad Y_p = 3e^{-2x} \]