1. (25 points) Consider the ODE $y'' + 5y' + 4y = 5e^x$.

(a) Please give the homogeneous solution to this ODE.

Homogeneous ODE: $y'' + 5y' + 4y = 0$

Characteristic Eqn: $\lambda^2 + 5\lambda + 4 = 0$

$(\lambda + 4)(\lambda + 1) = 0$

$\Rightarrow \lambda = -1, \lambda = -4$

Homogeneous solution: $y_h(x) = c_1 e^{-x} + c_2 e^{-4x}$

(b) Please give a particular solution to this ODE.

Undetermined Coefficients

$y_p(x) = Ae^x + C$

Plug in: $Ae^x + 5Ae^x + 4Ae^x = 5e^x$

$10Ae^x = 5e^x$

$A = \frac{1}{2}$

Particular solution: $y_p(x) = \frac{1}{2}e^x$

(c) Please give the homogeneous solution to this specific ODE.

General solution: $y(x) = y_h(x) + y_p(x) = c_1 e^{-x} + c_2 e^{-4x} + \frac{1}{2}e^x$
2. (20 points) Consider the ODE

\[(x^2 + 1)y' + xy = f(x)\]

\[\Rightarrow \quad y' + \frac{x}{x^2 + 1} y = \frac{f(x)}{x^2 + 1}\]

(a) (10 points) Please find an integrating factor \(\mu(x)\) for this equation. Simplify your answer as much as possible.

\[
\mu(x) = \frac{e^{\int \frac{x}{x^2 + 1} dx}}{e^{\frac{1}{2} \int \frac{2x}{x^2 + 1} dx}} = \frac{e^{\frac{1}{2} \int \frac{du}{u} = e^{\frac{1}{2} \ln u} = e^{\frac{1}{2} \ln (x^2 + 1)}}}{\text{Substitution integration}}
\]

\[\mu(x) = \sqrt{x^2 + 1}\]

(b) (10 points) Please find the general solution to this ODE. Your answer will have an arbitrary constant and depend on \(f(x)\).

\[
\begin{align*}
\sqrt{x^2 + 1} y' + \frac{x}{\sqrt{x^2 + 1}} y &= \frac{f(x)}{\sqrt{x^2 + 1}} \\
\frac{d}{dx} \left( \sqrt{x^2 + 1} y \right) &= \frac{f(x)}{\sqrt{x^2 + 1}} \\
\sqrt{x^2 + 1} y &= \int \frac{f(x)}{\sqrt{x^2 + 1}} dx + C
\end{align*}
\]

\[y(x) = \frac{1}{\sqrt{x^2 + 1}} \int \frac{f(x)}{\sqrt{x^2 + 1}} dx + C\]

3. (5 points) Suppose an electrical circuit contains a 5 ohm resistor, a 3 henry induction coil, a 0.1 farad capacitor, and a 1.5 volt battery (it's a AAA battery). Please write an ODE for the charge \(q\) in the circuit.

ODE: \[
3 \dddot{q} + 5 \dddot{q} + 10 \dot{q} = 1.5
\]
4. (15 points) Using any approach you wish, please compute the Laplace transform of \( f(x) = \cos x \). Hint: Euler Formula.

\[
\mathcal{L}\{\cos x\} = \frac{s}{s^2 + 1}
\]

5. (10 points) Suppose an isotope of a certain substance has a half-life of 15 minutes. If the initial amount of the isotope in a sample is \( m_0 \), how much of the isotope is left after 2 hours?

- Amount: \( \frac{m_0}{256} \)
6. (15 points) Consider the ODE:

\[ 2x^2 y'' + xy' - y = 0 \]

Please find the general solution of this ODE.

\[ \text{Ansatz: } y = x^\lambda \]

\[ 2x^2 \lambda (\lambda - 1) x^{\lambda - 2} + x \lambda x^{\lambda - 1} - x^\lambda = 0 \]

\[ (2\lambda^2 - 2\lambda) x^\lambda + \lambda x^\lambda - x^\lambda = 0 \]

\[ (2\lambda^2 - \lambda - 1) x^\lambda = 0 \]

\[ (2\lambda + 1)(\lambda - 1) = 0 \]

\[ \lambda_+ = 1 \]

\[ \lambda_- = \frac{1}{2} \]

General solution: \( y(x) = c_1 x^{-\frac{1}{2}} + c_2 x \)

7. (10 points) Suppose three functions \( u_1(t) \), \( u_2(t) \) and \( u_3(t) \) are linearly dependent on an interval \([c, b]\). Please give two distinct conditions that these three functions must satisfy.

1. \[ c_1 u_1(t) + c_2 u_2(t) + c_3 u_3(t) = 0 \]

\( \text{with at least one } c_k \neq 0. \)

2. \[ 0 = W[u_1, u_2, u_3] = \begin{vmatrix} u_1 & u_2 & u_3 \\ u_1' & u_2' & u_3' \\ u_1'' & u_2'' & u_3'' \end{vmatrix} = 0 \]