1. (20 points) Consider the implication $A \Rightarrow B$ where $A$ and $B$ are themselves statements. For this implication, please state the following:

(a) contrapositive: $
eg B \Rightarrow \neg A$

(b) negation: $A \wedge \neg B$

(c) direct implication: $A \Rightarrow B$

(d) converse: $B \Rightarrow A$

2. (20 points) **Theorem:** If a sequence $\{a_n\} \subseteq \mathbb{R}$ converges, then the limit is unique.

**Proof:** Suppose that $a_n \to a$ for some $a \in \mathbb{R}$ and that $a_n \to \alpha$ for some $\alpha \in \mathbb{R}$. To prove that the limit is unique, one must show that $a = \alpha$. So, given any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $|a_n - a| < \varepsilon/2$ and also $|a_n - \alpha| < \varepsilon/2$ for all $n > N$. By the triangle inequality,

$$|a - \alpha| \leq |a - a_n| + |a_n - \alpha| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Since $\varepsilon$ is arbitrarily small, $a = \alpha$ and thus the limit is unique. \qed
3. (15 points) Let \( f : A \rightarrow B \) be a function, and let \( S \subseteq B \) and \( T \subseteq B \). Please prove that \( f^{-1}(S \cap T) \subseteq f^{-1}(S) \cap f^{-1}(T) \).

Let \( x \in f^{-1}(S \cap T) \). Then \( f(x) \in S \cap T \), and thus \( f(x) \in S \) and \( f(x) \in T \). So \( x \in f^{-1}(S) \) and \( x \in f^{-1}(T) \) \( \Rightarrow \) \( x \in f^{-1}(S) \cap f^{-1}(T) \). All of this implies \( f^{-1}(S \cap T) \subseteq f^{-1}(S) \cap f^{-1}(T) \).

4. (15 points) Please show that

\[
\sum_{i=1}^{n} i = \frac{n(n+1)}{2}
\]

Hint: Induction.

- For \( n = 1 \), \( \sum_{i=1}^{1} i = 1 = \frac{1(1+1)}{2} \), so the formula works for this base case.

- Now assume the formula holds for \( n=k \); we must show it holds for \( n=k+1 \). Notice that

\[
\begin{align*}
\sum_{i=1}^{k+1} i &= \sum_{i=1}^{k} i + (k+1) \\
&= \frac{k(k+1)}{2} + (k+1) \\
&= \frac{k(k+1) + 2(k+1)}{2} \\
&= \frac{(k+2)(k+1)}{2} \\
&= \frac{(k+1)(k+1+1)}{2} \\
&= \frac{n(n+1)}{2}
\end{align*}
\]

Thus by induction, the formula holds \( \forall n \in \mathbb{Z}^+ \).
5. (15 points) Please negate the following statement about a sequence \( \{a_n\} \) and its limit \( a \):

\[
\forall N \in \mathbb{Z}^+, \exists \varepsilon > 0 \text{ s.t. } |a_n - a| < \varepsilon \forall n > N
\]

Negation: \( \exists N \in \mathbb{Z}^+ \text{ s.t. } \forall \varepsilon > 0, |a_n - a| \geq \varepsilon \) for some \( n > N \).

6. (15 points) Consider the sum of any six consecutive positive integers. What can said about the divisors (factors) of this sum? Please prove your answer.

Let \( S = n + (n+1) + (n+2) + (n+3) + (n+4) + (n+5) \) be the sum of six consecutive positive integers (depending on \( n \in \mathbb{Z}^+ \)). So \( S = 6n + 15 = 3(2n+5) \) and thus \( S \) is the product of two factors: 3 and an odd integer, 7 or greater.