Final, Part 1 (closed book, closed notes)  

1. (20 points) Please define/describe/state each of the following:

(a) Neumann Problem  
\[ \Delta u = 0 \quad x \in \Omega \]
\[ \partial_n u = 0 \quad x \in \partial \Omega \]

(b) Poisson Equation  
\[ \Delta u = f \]  
(c) Test Function  
\[ \psi \in C_c^\infty(\Omega) \]

(d) Similarity Solution  
For a PDE with two independent variables, a solution where these variables appear only in a certain combination, e.g.,  
\[ f = \frac{x}{t^2} \]

2. (10 points) Suppose a harmonic function \( u \) is defined on a bounded domain \( \Omega \). What does the maximum principle guarantee?  
That  \( \max_{x \in \Omega}(u) \subset \partial \Omega \) and  \( \min_{x \in \Omega}(u) \subset \partial \Omega \) iff \( u \) is constant on \( \overline{\Omega} \).

3. (20 points) Consider the PDE \( a(x, y)u_x + b(x, y)u_y = 0 \). According to the chain rule, what equations involving a variable \( s \) must be satisfied if \( u \) is to be constant with respect to \( s \)? Explain your answer.

By the chain rule, we need  
\[ \frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial s} = a(x, y)u_x + b(x, y)u_y = 0 \]

Thus \( u \) is constant w.r.t. \( s \) only if

Equations:  
\[ \frac{\partial x}{\partial s} = a(x, y) \]
\[ \frac{\partial y}{\partial s} = b(x, y) \]
Final, Part 2 (open book, open notes)  D Term, 2005

Show all work needed to reach your answers. You may use any theorem we discussed in the course, but please cite any result you use.

1. (25 points) Consider the following heat-flow system:

PDE: \( u_t = \kappa \Delta u \quad x \in \Omega, \ t > 0 \)

BC: \( u(x, t) = 0 \quad x \in \partial \Omega, \ t > 0 \)

IC: \( u(x, 0) = u_0(x) \quad x \in \Omega \)

where \( \kappa > 0 \) is constant and \( u_0 \) is a given function. Suppose that energy for this system is defined as

\[
E(u) := \frac{1}{2} \int_{\Omega} u^2 \, dV
\]

where the integral represents a multiple integral. Please show that \( E \) is decreasing.

\[
\frac{dE}{dt} = \frac{1}{2} \int \frac{\partial}{\partial t} (u^2) \, dV = \int uu_t \, dV = \int (\kappa u \Delta u) \, dV = \kappa \left( \int u \, ds - \int \kappa u^2 \, dV \right)
\]

\[
= -\kappa \int \kappa u^2 \, dV \leq 0
\]

\( \bullet \) The maximum value of this energy is \( E(0) = \frac{1}{2} \int u_0^2 \, dV \).

\( \bullet \) Notice that \( \frac{dE}{dt} < 0 \) unless \( u \) is constant which means \( u_0 \) given the BC.
2. (25 points) For $x \in \mathbb{R}^3$, please give the solution (in symbols) of the following wave problem:

\[
\begin{align*}
PDE: \quad \Box u &= 0 \quad x \in \mathbb{R}^3, \ t > 0 \\
\quad u(x, 0) &= 0 \quad x \in \mathbb{R}^3 \\
\quad \text{IC: } u_t(x, 0) &= 1 \quad |x| = 1 \\
\quad u_t(x, 0) &= 0 \quad |x| \neq 1
\end{align*}
\]

So if $u$ represents light, the initial conditions are zero everywhere except on the spherical shell with radius 1 centered at the origin where at time $t = 0$ there is a flash (nonzero time derivative of $u$). Please describe in words what the solution looks like.

This is a really interesting question. One answer is to note that $u(x, 0) = 0$ and $u_t(x, 0) = 0$ a.e., so $u(x, t) = 0$ a.e. for $x \in \mathbb{R}^3$ and $t > 0$. But this is probably not the best solution.

Another approach is to note that by (3), §9.2 (p. 224) Strassen

\[
\frac{1}{4\pi t} \int_{|x-y|=t} \zeta_s(x) \, dS
\]

where $C=1$, $\phi \equiv 0$ and $\zeta_s(x)$ denotes the spherical shell with radius 1 centered at the origin. Thus for $x=0$ and $t=1$, the integral yields the surface area of $\zeta_s(0)$, so $u(0,1) = 1$. But for all other values of $x$ and $t$, the domain of integration intersects $\zeta_s(x)$ only on a curve (circle) or at a single point. Since the area under a circle or a point is zero, $u(x,t) = 0$ unless $x=0$ and $t=1$. 
