Show all work needed to reach your answers.

1. (20 points) The equation \( u_{tt} - c^2 \Delta u + m^2 \sin(u) = 0 \) is called the Sine-Gordon equation. What is the energy? Is it constant (prove or disprove)?

2. (20 points) Please describe and solve the first order PDE \( u_x + u_y = u \) subject to the auxiliary condition \( u(x, 0) = \cos x \). (Hint: you can get substantial credit for describing the solution even if you can not find it in closed form. A sketch of the \( x, y \)-plane may help.)

3. (20 points) Please find the harmonic function \( u \) on the disk \( \Omega \subset \mathbb{R}^2 \) with radius \( r = 1 \) which satisfies \( u = 1 + \cos \theta \) on \( \partial \Omega \). Is this function \( u \) unique (prove or disprove)?

1. For smooth solutions, if \( \frac{\partial^2 u}{\partial t^2} - c^2 \Delta u + m^2 \sin(u) = 0 \), then

\[
0 = \left( \frac{\partial^2 u}{\partial t^2} - c^2 \Delta u + m^2 \sin(u) \right) U_t = \frac{\partial}{\partial t} \left( \frac{\partial^2 u}{\partial t^2} + c^2 \left| \nabla u \right|^2 - m^2 \cos(u) \right) - c^2 \Delta u \left( \frac{\partial u}{\partial t} \right)_t
\]

Integrating over all of \( \mathbb{R}^n \) and assuming that \( u \) goes to zero far from the origin, one finds that \( \dot{E}(t) = \int_{\Omega} \frac{\partial^2 u}{\partial t^2} \left( \frac{\partial^2 u}{\partial t^2} + c^2 \left| \nabla u \right|^2 - m^2 \cos(u) \right) \, dV \) satisfies

\[
\dot{E}(t) = \int_{\Omega} \frac{\partial^2 u}{\partial t^2} \left( \frac{\partial^2 u}{\partial t^2} + c^2 \left| \nabla u \right|^2 - m^2 \cos(u) \right) \, dV = -\int_{\Omega} c^2 \Delta u \left( \frac{\partial u}{\partial t} \right)_t \, dV = 0.
\]

2. Since \( \frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} U_x + \frac{\partial u}{\partial y} U_y = u \) if \( \frac{\partial u}{\partial x} = 1 \) and \( \frac{\partial u}{\partial y} = 1 \), the characteristics require \( x = \xi + f(y) \) and \( y = \xi + g(x) \). Since the \( x \)-direction relative to the \((x,y)\)-plane is \((1, 0)\), the perpendicular \( y \)-direction is \((-1, 1)\), the simplest choice is \( \xi = \frac{x+y}{2} \) and \( \eta = \frac{x-y}{2} \).

Now the solution of \( \frac{\partial u}{\partial t} = u \) is \( u = A(\xi) e^{\frac{\xi}{2}} = A(\frac{x-y}{2}) e^{\frac{x+y}{2}} \).

Because \( u(x, 0) = \cos(x) = A(\frac{1}{2}) e^{\frac{1}{2}} \), then \( A(\frac{1}{2}) = e^{-\frac{1}{2}} \cos(x) \), or \( A(\frac{1}{2}) = e^{\frac{1}{2}} \cos(2x) \).

And

\[
u(x, t) = e^{\frac{y-x}{2}} \cos(x-y) e^{\frac{x+y}{2}} = e^y \cos(x-y)
\]
Suppose \( u, v \), two harmonic functions on \( \Omega \) satisfying \( u = v = 1 + \cos \theta \) on \( \partial \Omega \). Then \( w = u - v \) is harmonic and \( w = 0 \) on \( \partial \Omega \).

By the maximum principle, \( w \equiv 0 \) on \( \Omega \).

So \( u = v \implies \) the solution must be unique.

What's the solution? Notice that since \( \partial \Omega = \{ (x, y) \mid y = 1 \} \), on this boundary \( u = 1 + \cos \theta = 1 + x \cos \theta = 1 + x \).

Since \( u(x, y) \equiv 1 + x \) is harmonic, this must be the unique solution.